Dealing with Uncertainty

in Decision Making Models

Johannes O. Royset & Roger J-B Wets

joroyset@nps.edu

Naval Postgraduate School & U. Callifornia, Davis

I. A product mix problem

A formulation

A furniture manufacturer must choose $x_j \geq 0$, how many dressers of type $j = 1, \ldots, 4$ to manufacture so as to maximize profit

$$\sum_{j=1}^{4} c_j x_j = 12x_1 + 25x_2 + 21x_3 + 40x_4$$

The constraints:

$$t_{c1}x_1 + t_{c2}x_2 + t_{c3}x_3 + t_{c4}x_4 \leq d_c$$
$$t_{f1}x_1 + t_{f2}x_1 + t_{f3}x_1 + t_{f4}x_1 \leq d_f$$

t_{cj} (t_{fj}) carpentry (finishing) man-hours: dresser type j

d_c (d_f) = total time available for carpentry (finishing)

Product mix problem (2)

Solution via linear programming:

$$\max\langle c, x\rangle \text{ so that } Tx \le d,\ x \in I\!R^n_+.$$

With

$$T = \begin{bmatrix} t_{c1} & t_{c2} & t_{c3} & t_{c4} \\ t_{f1} & t_{f2} & t_{f3} & t_{f4} \end{bmatrix} = \begin{bmatrix} 4 & 9 & 7 & 10 \\ 1 & 1 & 3 & 40 \end{bmatrix}, \quad \begin{bmatrix} d_c \\ d_f \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$$

Optimal: $\boxed{x^d = (4000/3,\ 0,\ 0,\ 200/3)}$

Value: $ 18,667.

Product mix problem (3)

But . . . "reality" can't be ignored!

$$\boldsymbol{t}_{cj} = t_{cj} + \boldsymbol{\eta}_{cj}, \quad \boldsymbol{t}_{fj} = t_{fj} + \boldsymbol{\eta}_{fj}$$

entry	possible values			
$d_c + \zeta_c$:	5,873	5,967	6,033	6,127
$d_f + \zeta_f$:	3,936	3,984	4,016	4,064

10 random variables, say, 4 possible values each

$L = 1,048,576$ possible pairs (T^l, d^l)

Product mix problem (4)

What if $\sum_{j=1}^{4}(t_{cj} + \eta_{cj})x_j > d_c + \zeta_c$? $\implies$ overtime

With $\xi = (\eta_{\{\cdot,\cdot\}}, \zeta_{\{\cdot\}})$, **recourse**: $(y_c(\xi), y_f(\xi))$ @ cost (q_c, q_f).

$$\begin{array}{lllllll}
\max & \langle c, x\rangle & -p_1\langle q, y^1\rangle & -p_2\langle q, y^2\rangle & \cdots - & p_L\langle q, y^L\rangle & & \\
\text{s.t.} & T^1 x & -y^1 & & & & \leq & d^1 \\
 & T^2 x & & -y^2 & & & \leq & d^2 \\
 & \vdots & & & \ddots & & & \vdots \\
 & T^L x & & & & -y^L & \leq & d^L \\
 & x \geq 0, & y^1 \geq 0, & y^2 \geq 0, & \cdots & y^L \geq 0. & &
\end{array}$$

Structured large scale l.p. $(L \approx 10^6)$

Product mix problem (5)

Define $\Xi = \{\xi = (\eta, \zeta)\}$, $p_\xi =$ prob $[\boldsymbol{\xi} = \xi]$

$$Q(\xi, x) = \max \left\{ \langle -q, y \rangle \,\middle|\, T_\xi x - y \le d_\xi,\ y \ge 0 \right\}$$

$$EQ(x) = E\{Q(\boldsymbol{\xi}, x)\} = \sum_{\xi \in \Xi} p_\xi Q(\xi, x)$$

the equivalent deterministic program (DEP):

$$\max \langle c, x \rangle + EQ(x) \;\; \text{so that} \;\; x \in \mathbb{R}^n_+.$$

a *non-smooth* convex optimization problem: EQ concave.

Product mix problem (6)

Solution of DEP, or large scale l.p.,:

Optimal: $\boxed{x^* = (257,\ 0,\ 665.2,\ 33.8)}$

expected Profit: $ 18,051

The solution x^* is *robust* : it considered all $\approx 10^6$ possibilities.

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Recall: $x^d = (1333.33,\ 0,\ 0,\ 66.67)$

expected "profit" relying on x^d = $ 16,942.

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- x^d is not close to optimal
- x^d isn’t pointing in the right direction

Mathematics & Numerics

Stochastic Programming relies on:

- linear, non-linear, mixed-integer programming
- large scale: decomposition methods, structured programs, grid computing
- Variational Analysis: non-smooth, duality, epi-convergence (approximations), etc.
- Probability: stochastic processes, asymptotic laws
- Statistics: estimation, lack of data issues
- Functional Analysis, Combinatorial Geometry, etc.

II. Modeling, modeling & modeling!

Uncertain parameters

Deterministic Optimization problem:

$$\min f_0(x) \text{ so that } x \in S \subset I\!R^n$$

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Uncertain parameters: $\xi \in \Xi \subset I\!R^N$,

$$\min f_0(\xi, x) \;\text{ so that }\; x \in S(\xi) \subset I\!R^n$$

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Wait-and-see solution ??

$$x(\xi) \in \text{argmin}\left\{ f_0(\xi, x) \,\middle|\, x \in S(\xi) \right\}$$

What's needed: a *here-and-now* solution.

The NewsVendor Problem

- $\xi \in \Xi \subset \mathbb{R}_+$ demand for a (perishable) good
  e.g., plant capacity, overbooking, etc.
- $x \geq 0$ quantity ordered @ unit cost: $c = 10$
- $y \geq 0$ quantity sold, per unit profit $r = 15$

Total revenue (possibly negative):

$$-cx + (c+r)y \text{ where } 0 \leq x, \quad 0 \leq y \leq \min\{x, \xi\}$$

Find optimal x^*!

The “deterministic” approach

Pick $\hat{\xi} \in \Xi$ (guessing the future) and solve

$$\min f_0(\hat{\xi}, x) \;\; \text{so that} \;\; x \in S(\hat{\xi}) \subset I\!R^n$$

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NewsVendor: $\Xi = [0, 150]$, pick $\hat{\xi} = 75$,

$$\max -cx + (c + r)y$$
$$x \geq 0, \quad 0 \leq y \leq \min\{x, \hat{\xi}\}$$

Solution: $x^o = y^o = \hat{\xi}$, obj. value = $r\hat{\xi} = 1125$

But doesn’t tell much about “profit” if $\xi \neq 75$!

Scenario Analysis

Pick $\xi^1, \ldots, \xi^L$ (scenarios), and for each ξ^l find:

$$x^l \in \operatorname{argmin} \left\{ f_0(\xi^l, x) \,\middle|\, x \in S(\xi^l) \right\}$$

and “reconcile” the solutions to obtain x^o.

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and "reconcile" the solutions to obtain x^o.

NewsVendor: pick $\xi^1 = 10, \xi^2 = 20, \ldots, \xi^{15} = 150$,

$$(x^l, y^l) \in \underset{x \geq 0, y \geq 0}{\text{argmax}} \{-cx + (c+r)y \,|\, y \leq \min[\xi^l, x]\}$$

Wait-and-see sol'ns: $x^l = \xi^l$. "Reconciliation"?

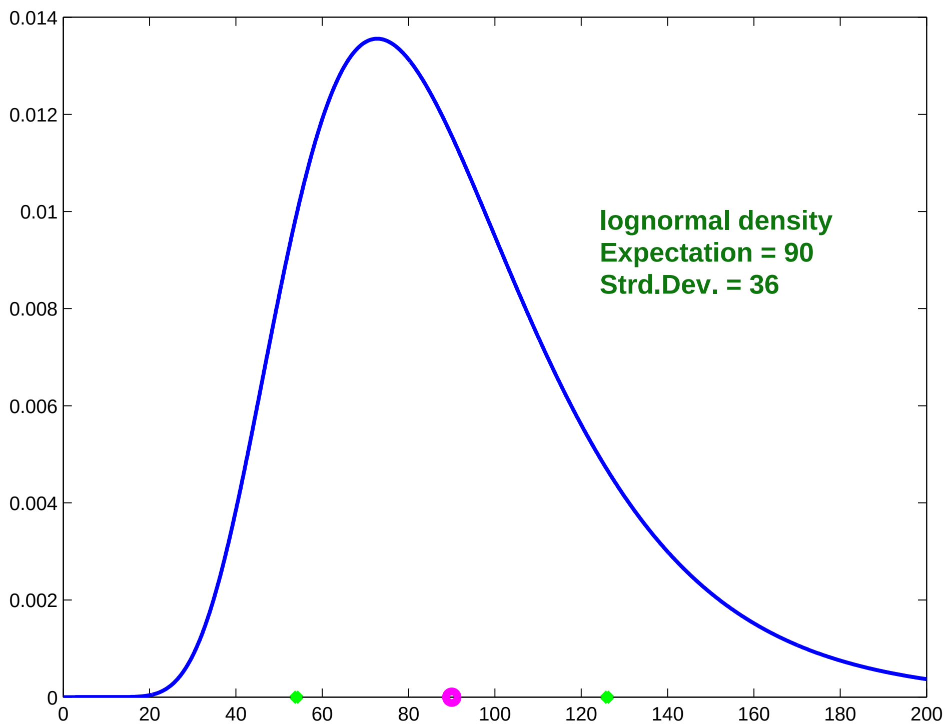
No help in choosing x^o the quantity to order.

ξ: Estimated Density h

ξ log-normal: $h(z) = (z\tau\sqrt{2\pi})^{-1}\, e^{-\frac{(\ln z - \theta)^2}{2\tau^2}}$

$\theta = 4.43, \tau = 0.38;\ \ H(z) = \int_0^z h(s)\, ds$

from data, expert(s), all information available



might affect choice of $\hat{\xi}$, scenarios: $\xi^1, \ldots$

Maximize Expected Return

$$\begin{aligned} \max \quad & -cx + E\{(c+r)y_\xi\} \\ \text{so that } & x \geq 0,\ 0 \leq y_\xi \leq \min[\xi, x] \end{aligned}$$

The *equivalent deterministic program*:

$$\max_{x \geq 0} -cx + EQ(x), \quad EQ(x) = E\{Q(\boldsymbol{\xi}, x)\}$$

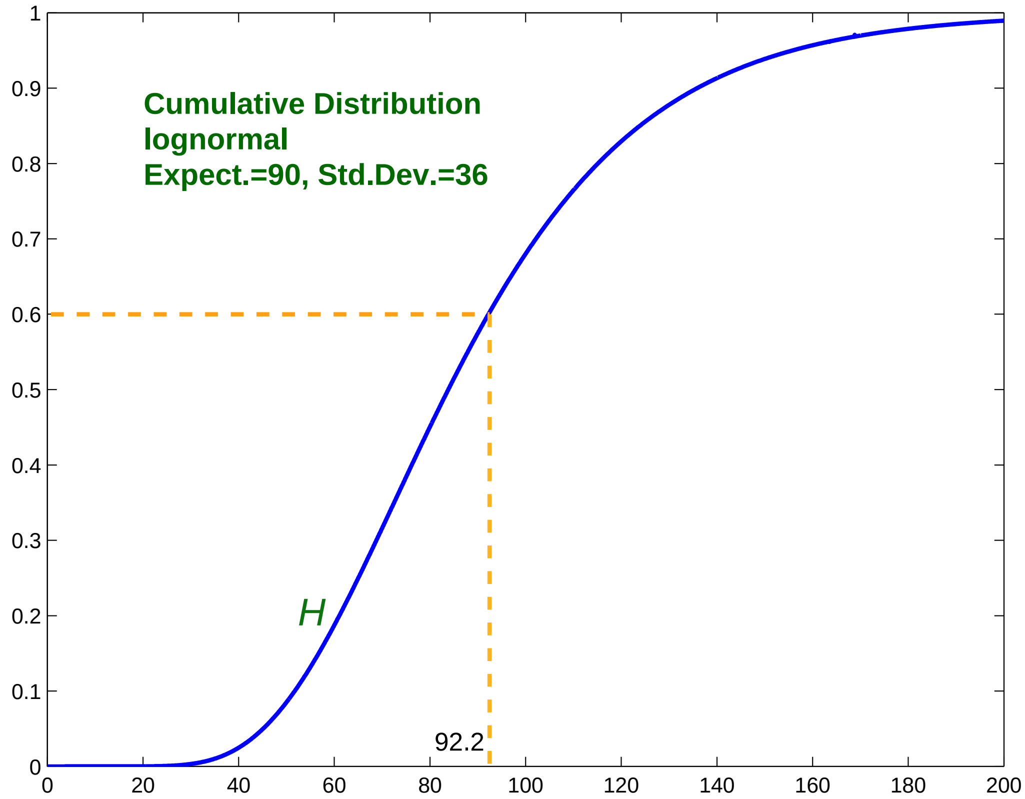
where $Q(\xi, x) = \begin{cases} (c+r)\xi & \text{if } \xi \leq x, \\ (c+r)x & \text{if } \xi \geq x \end{cases}$

$$EQ(x) = (c+r)\left(\int_0^x \xi\, H(d\xi) + \int_x^\infty x\, H(d\xi)\right)$$

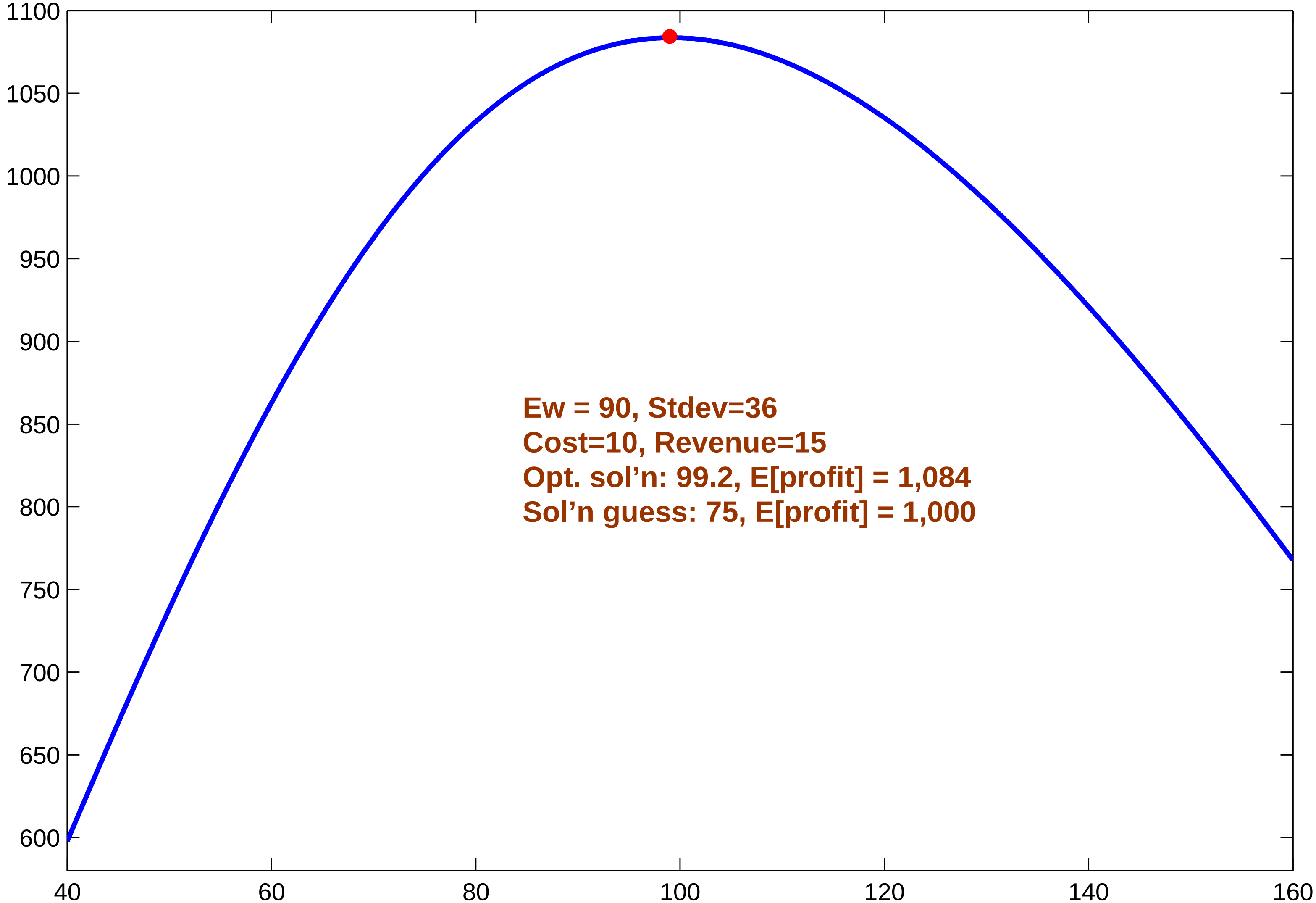
Optimal: Expected Profit

$$x^* = H^{-1}\left(\frac{r}{c+r}\right) = H^{-1}(0.6) = 99.2$$

for $c = 10, r = 15$.

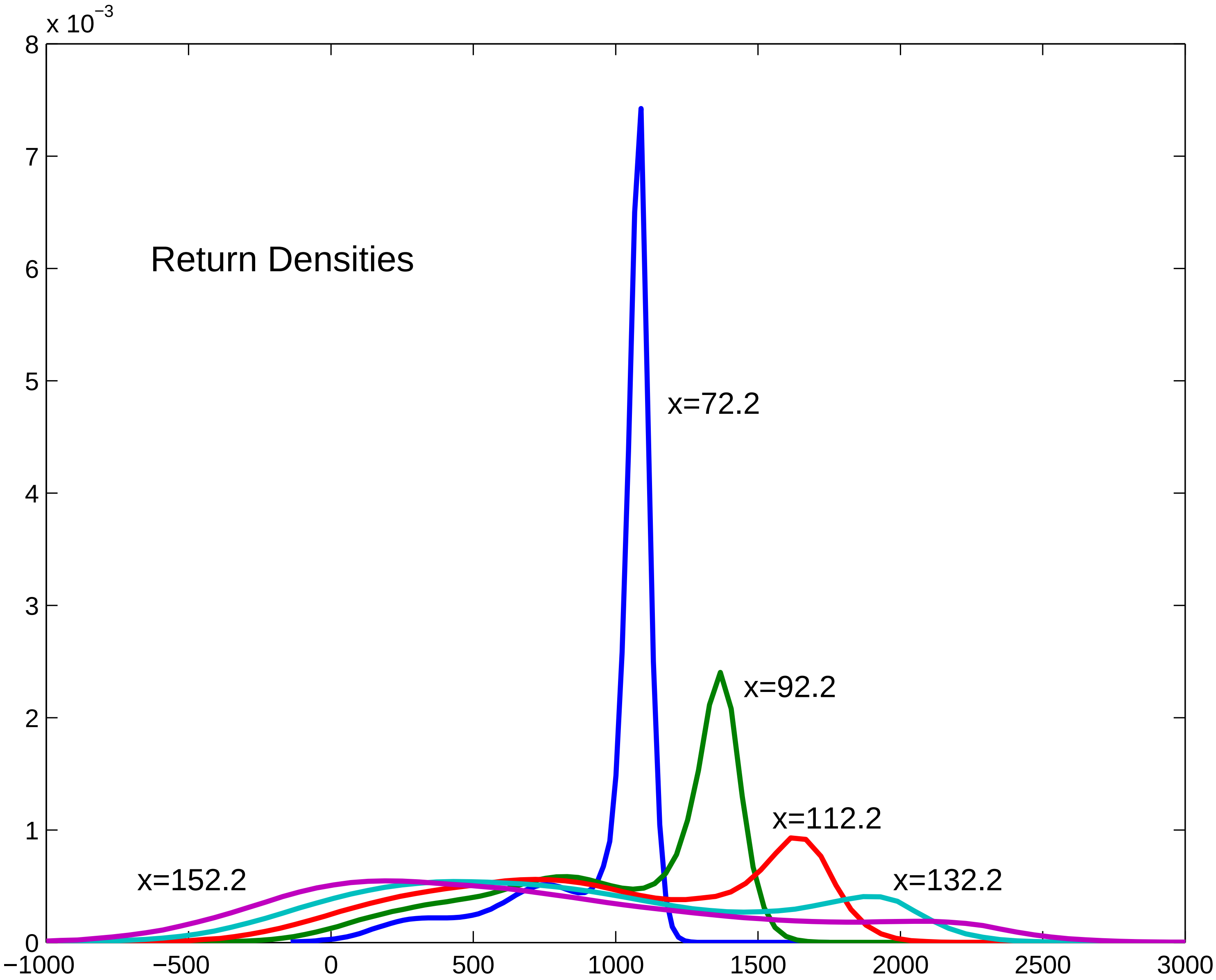


NewsVendor's Objective

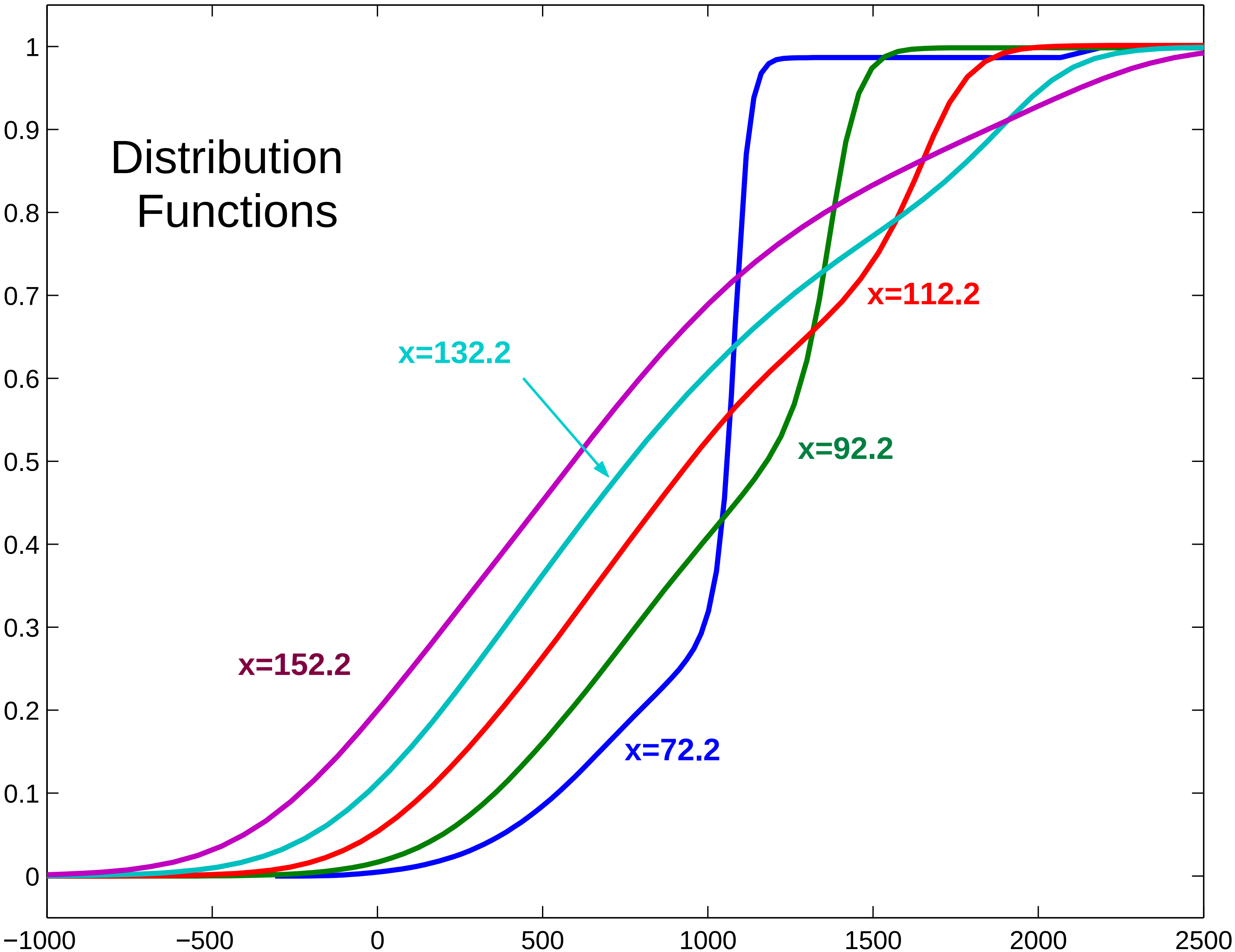


. . . but is maximum expected return the “real” objective?

The Returns' Densities



Choosing the Returns’ Distribution



Decision Criteria

Reducing the choice of a distribution function
to the choice of a “number”

- maximize expected return (scaled?),
- max. E{return} & minimize customers lost,
- minimize Value-at-Risk (VaR),
- minimize the (buffered) probability of failure,
- minimizing a Measure
- variants & combinations of the above

Maximizing Expected Utility

generic" stochastic optimization problem:

$$\max E\{f_0(\xi, x)\} \text{ such that } \quad f_i(\xi, x) \le 0, \; i = 1, \ldots, m,$$

Risk-averse or risk-seeking $\Longrightarrow$ *utility function*

von Neuman-Morgenstern: under "rationality" axiomatics) there exists a utility function u such hat $\bar{x} \in \operatorname{argmax} E\{u\big(f_0(\xi, x)\big)\}$ (subject to the constraints) identifies the preferred return's distribution

Modeling hurdle: no blueprint for u's design!

Robust Optimization

"generic" optimization problem: $\max \ \gamma$

$$\text{so that } \gamma - f_0(\xi, x) \leq 0,$$
$$f_i(\xi, x) \leq 0, \ i = 1, \dots, m,$$

"robust" counterpart: $\max \ \gamma$

$$\text{so that } \gamma - f_0(\xi, x) \leq 0, \ \forall \xi \in \mathcal{U} \subset \Xi$$
$$f_i(\xi, x) \leq 0, \ i = 1, \dots, m, \ \forall \xi \in \mathcal{U},$$

Challenges:

- formulate a computationally tractable robust counterpart
- specify reasonable uncertainty for set $\mathcal{U}$

Reliability: Chance Constraints

Satisfy constraints with probability $\alpha \in (0,1]$

$$\min f_0(x) \text{ so that } \text{prob.}\,[\, x \in S(\boldsymbol{\xi})\,] \geq \alpha$$

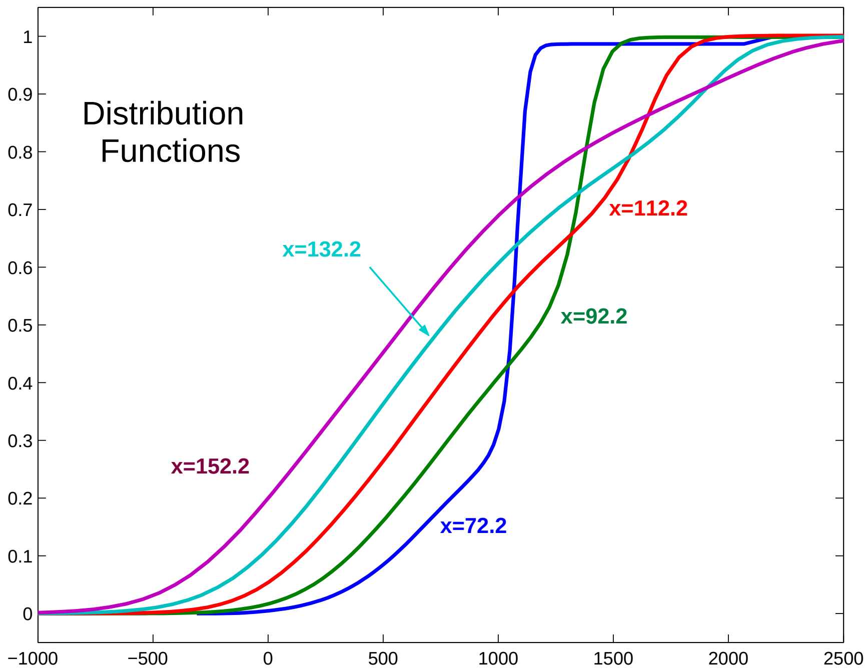
Variant:

$$\begin{aligned} \min \quad & f_0(x) \\ \text{so that} \quad & \text{prob.}\,[\, f_i(\boldsymbol{\xi}, x) \leq 0\,] \geq \alpha_i,\ i \in I \end{aligned}$$

α_i dictated by

- contractual obligations, company policy, guess, etc.
- often nonconvex; convex alternative, buffered prob.

Stochastic Dominance



Stochastic Dominance: $D_x(s) \leq D_{\hat{x}}(s), \quad \forall\, s$

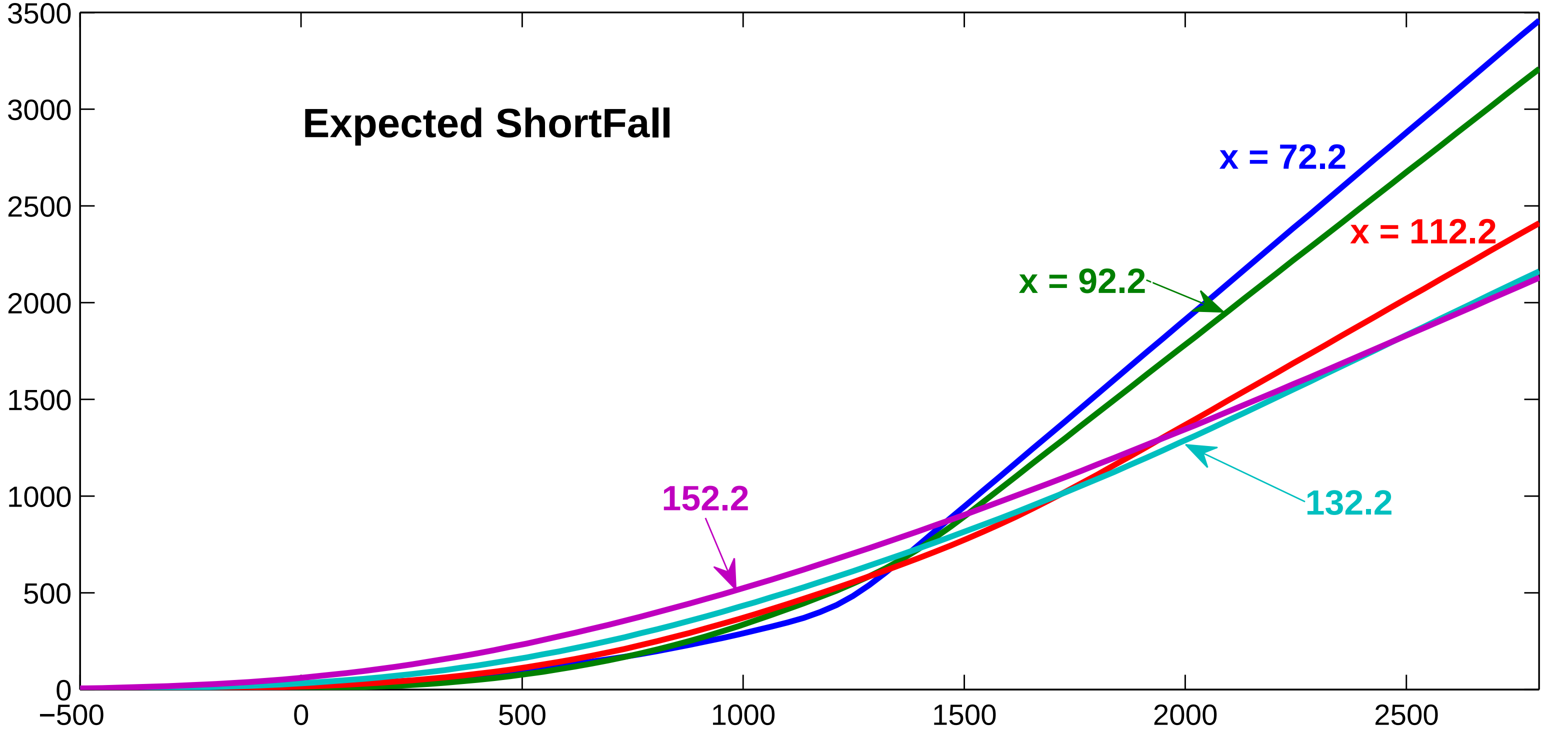
$\Longrightarrow$ probability of the return to be $\leq s$ always smaller when choosing x rather than $\hat{x}$

unfortunately unusual

Second order Stochastic Dominance

$$D^2(s) = \int_{-\infty}^{s} D(\xi)\, d\xi = E\{(s - \boldsymbol{\xi})_+\}$$

D^2: the expected shortfall function



Stochastic Dominance Constraint

NewsVendor problem

$$\max rx, \quad x \geq 0$$
$$\text{such that } D_x^2(s) \leq G^2(s), \quad s \in [\alpha, \beta]$$

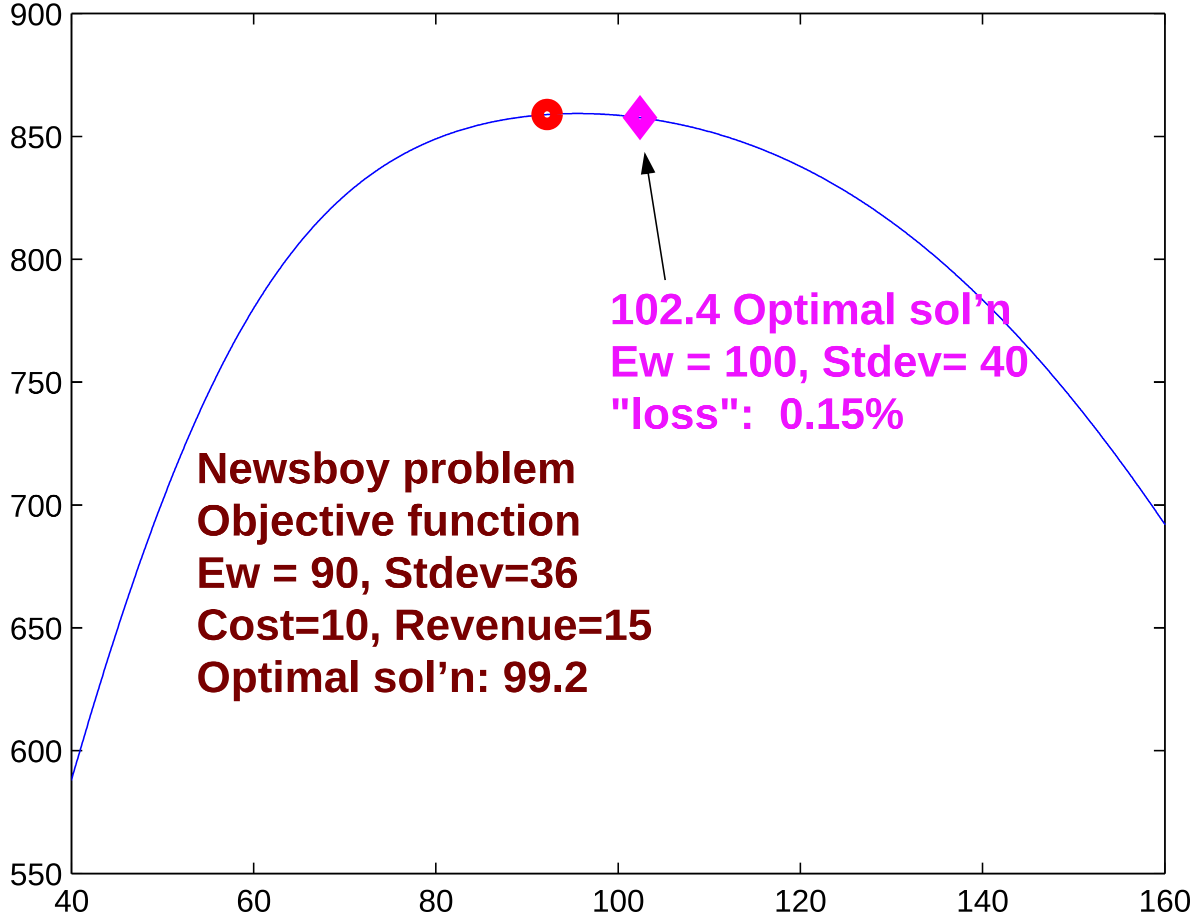
given a “desirable” distribution function G

D_x: distribution of actual return, decision x

rx when $\xi \leq x$ and $(c+r)\xi - cx$ when $\xi < x$

leads to a semi-infinite optimization problem used in portfolio optimization, for example

Perturbing the Probability Measure



stress testing via distribution contamination

A few references - Books

1. A. Shapiro, D. Dentcheva & A. Ruszczynski, *Lectures on Stochastic Programming*, MPS-SIAM Series. 2009.
2. G. Pflug & A. Pichler, Multistage Stochastic Programming, Springer, 2014.
3. A. King & S. Wallace, Modeling with Stochastic Programming, Springer, 2012.
4. G. Pflug & W. Römisch, *Modeling, Measuring and Managing Risk*, World Scientific, 2007

A few references - Tutorials/Surveys

1. Excellent tutorials at stoporg.org
2. Upcoming tutorial by A. Shapiro and previous tutorial by J. Luedtke
3. Overview of Risk-averse models:
   R.T. Rockafellar and J. Royset,
   Engineering Decisions under Risk-Averseness,
   ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering, Vol. 1, No. 2, 2015.