Optimization Problems with Probabilistic Constraints

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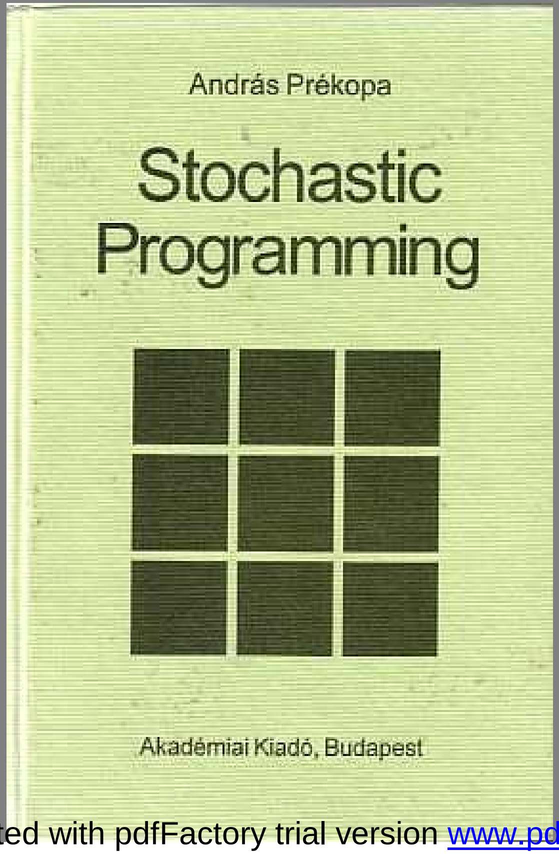
Recommended Reading

A. Prekopa:
Stochastic Programming
Kluwer, Dordrecht, 1995.

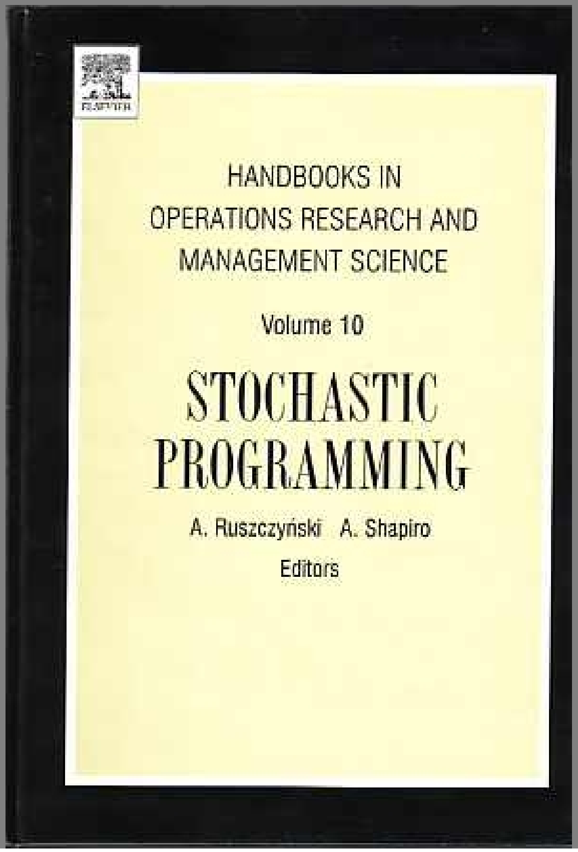


András Prékopa

Stochastic Programming

Akadémiai Kiadó, Budapest

A. Prekopa:
Probabilistic Programming
Chapter 5 in A. Ruszczynski and A. Shapiro (eds.):
Stochastic Programming.
Handbooks in Operations Research and
Management Science, Vol. 10,
Elsevier, 2003.Amsterdam



HANDBOOKS IN
OPERATIONS RESEARCH AND
MANAGEMENT SCIENCE

Volume 10

STOCHASTIC
PROGRAMMING

A. Ruszczyński A. Shapiro
Editors

Overview

1. Example
2. Models
3. Structure
4. Numerics
5. Stability

Probabilistic Constraints

conventional optimization problem: $min\{f(x)|h_j(x) \geq 0 \ \ (j=1,...,m)\}$

In many real life problems from finance, engineering etc., the constraints involve random parameters due to demographic, meteorological, economical etc. uncertainties: $h_j(x,\xi) \geq 0$

Often, a decision x has to be taken before ξ is observed ('here and now').

No matter, how x is chosen, there is no guarantee that $h_j(x,\xi) \geq 0$ for all possible realizations of ξ.

If the distribution of ξ is known, we may calculate for each x the probability of constraint satisfaction: $P(h_j(x,\xi) \geq 0 \ \ (j=1,...,m))$

reasonable: x feasible if this probability is larger than some safety level p.

optimization problem with probabilistic constraints::

$$min\{f(x)|P(h_j(x) \geq 0 \ \ (j=1,...,m)) \geq p\} \ \ (p\in[0,1])$$

Applications: finance, power generation, water managment, telecommunication, chemical engineering etc.

1. Example

The cash matching problem[1]

The pension fund of a company has to make payments for the next 15 years.

Payments shall be covered by investing an initial capital K in bonds of

3 different types.

Decision variables: x_1, x_2, x_3 - number of bonds of each type to be bought.

Objective: Maximize final amount of cash (after 15 years)

Constraints: cover payments in all years.

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[1] A. Ruszczynski (www.rusz.rutgers.edu) and Dentcheva, Lai, Ruszczynski (2003)

Data and liquidity constraints

year	payments	yields per bond of type 1	2	3
1	11,000	0	0	0
2	12,000	60	65	75
3	14,000	60	65	75
4	15,000	60	65	75
5	16,000	60	65	75
6	18,000	1060	65	75
7	20,000	0	65	75
8	21,000	0	65	75
9	22,000	0	65	75
10	24,000	0	65	75
11	25,000	0	65	75
12	30,000	0	1060	75
13	31,000	0	0	75
14	31,000	0	0	75
15	31,000	0	0	1075
		cost per bond:		
		980	970	1050

α_{ij}

β_j

γ_i

K = 250,000 (initial capital)

cash available at the end of year j:

$$\underbrace{K-\sum_{i=1}^{3}\gamma_i x_i}_{\text{cash after buying bonds}} + \underbrace{\sum_{k=1}^{j}\sum_{i=1}^{3}\alpha_{ik}x_i}_{\text{cumulative yields of bonds}} - \underbrace{\sum_{k=1}^{j}\beta_k}_{\text{cumulative payments}} \geq 0$$

Rearrangement:

$$a_{ij} := \sum_{k=1}^{j}\alpha_{ik}-\gamma_i \qquad b_j := \sum_{k=1}^{j}\beta_k - K$$

→ liquidity constraints:

$$\sum_{i=1}^{3} a_{ij}x_i - b_j \geq 0 \quad (j=1,\dots,15)$$

Linear optimization problem

liquidity in year j: $\sum_{i=1}^{3} a_{ij} x_i - b_j \geq 0$

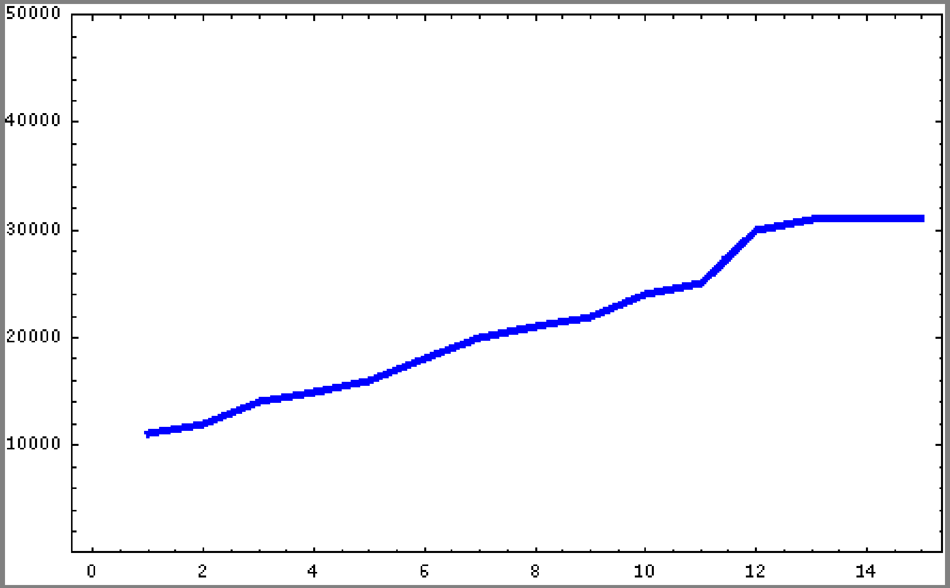
final cash: $\sum_{i=1}^{3} a_{i,15} x_i - b_{15}$

optimization problem: $max \left\{ \sum_{i=1}^{3} a_{i,15} x_i \mid \sum_{i=1}^{3} a_{ij} x_i \geq b_j \ (j=1,...,15) \right\}$

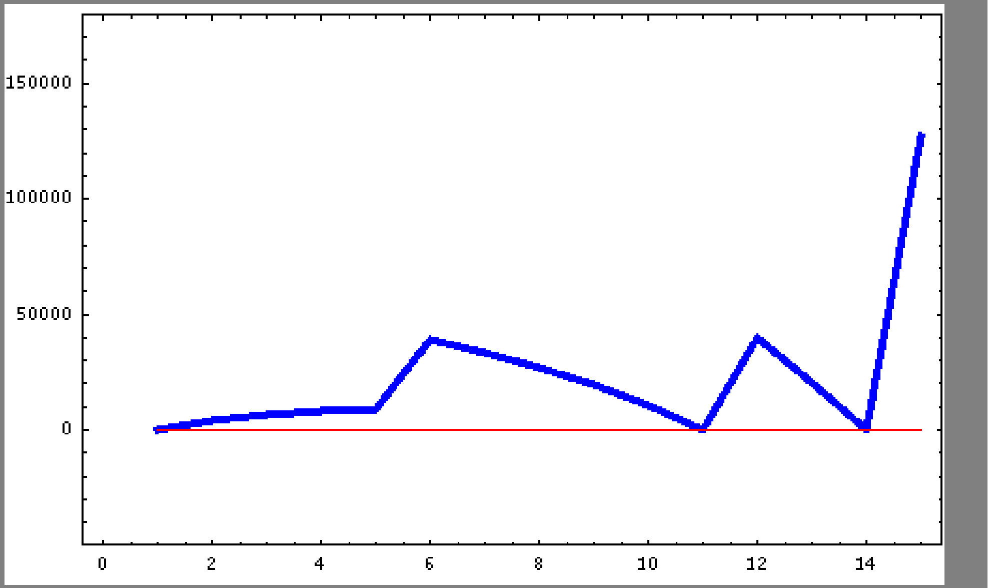
Solution of the deterministic cash matching problem

$$x^{opt}=\{31.1\,,55.5\,,147.3\}\ \Rightarrow\ f\left(x^{opt}\right)=127{,}332$$

payments

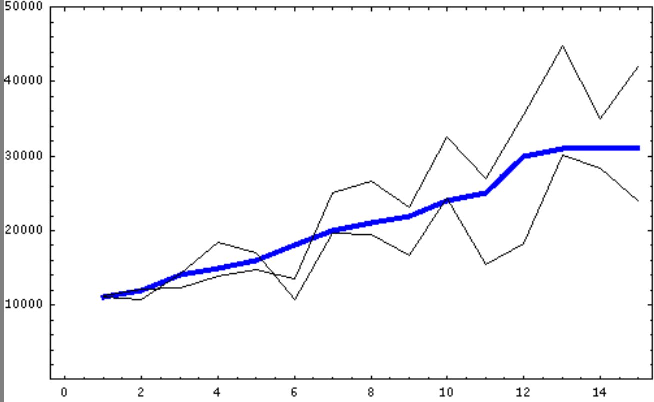


cash

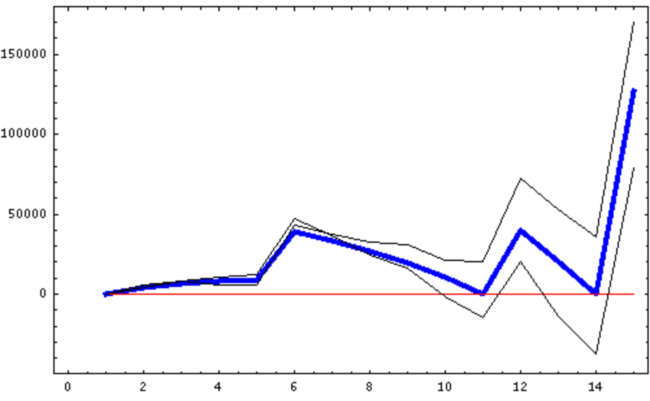


Random Payments

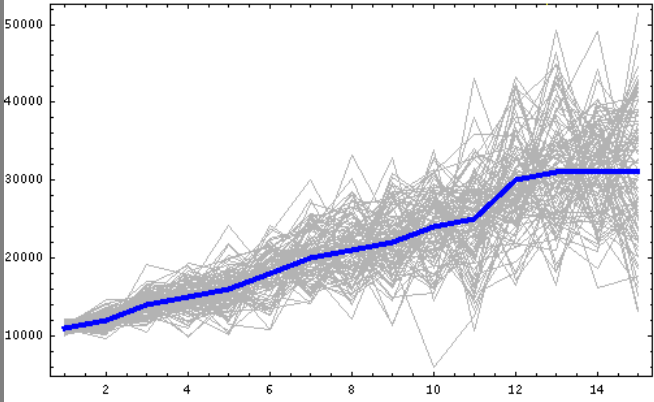
payments (expected value + 2 scenarios)



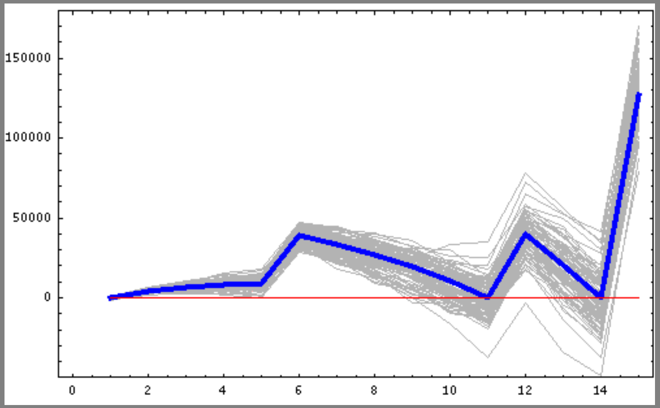
cash (expected value + 2 scenarios)



payments (expected value + 100 scenarios)

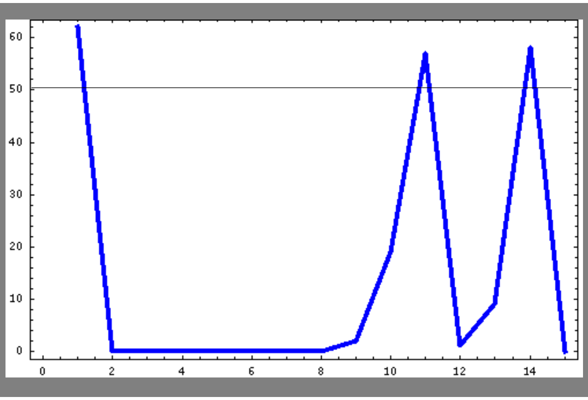


cash (expected value + 100 scenarios)



number of scenarios violating positive cash
(empirical probability of constraint violation)

50 %



Model with individual probabilistic constraints

Now, payments are random variables η_j

Assumption: independent, normal distribution,

$$E\,\eta_j \;=\; \beta_j, \qquad \sqrt{Var\,\eta_j} \;=\; 500\,j$$

expected payments =
deterministic payments

variance increasing with time

cumulative payments: $\xi_j \;:=\; \sum_{k=1}^{j} \eta_k \longrightarrow$ normal distribution

$$E\,\xi_j \;=\; b_j, \qquad Var\,\xi_j \;=\; \sum_{k=1}^{j} Var\,\eta_j$$

optimization problem with random parameter:

$$max\;\left\{\; \sum_{i=1}^{3} a_{i,15}\,x_i \;\middle|\; \sum_{i=1}^{3} a_{ij}\,x_i \;\geq\; \xi_j \;\; (j=1,\ldots,15) \;\right\}$$

Difficulty: decide on x before ξ is observed ('here-and-now' decision)

(continued)

expected-value problem (= our deterministic problem):

$$max \ \{ \ \sum_{i=1}^{3} a_{i,15} x_i \ | \ \sum_{i=1}^{3} a_{ij} x_i \ \geq \ E\,\xi_j \ = \ b_j \ \ (j=1,\ldots,15) \ \}$$

probabilistic constraint: $P(\sum_{i=1}^{3} a_{ij} x_i \ \geq \ \xi_j) \ \geq \ p \ \ (\in[0,1])$

probability measure

probability level

optimization problem with individual probabilistic constraints:

$$\boxed{max \ \{ \ \sum_{i=1}^{3} a_{i,15} x_i \ | \ P\left(\sum_{i=1}^{3} a_{ij} x_i \ \geq \ \xi_j\right) \ \geq \ p \ \ (j=1,\ldots,15) \ \}}$$

(continued)

$$P\left(\sum_{i=1}^{3} a_{ij} x_i \geq \xi_j\right) = P\left(\frac{\sum_{i=1}^{3} a_{ij} x_i - E(\xi_j)}{\sqrt{Var(\xi_j)}} \geq \underset{\uparrow}{\frac{\xi_j - E(\xi_j)}{\sqrt{Var(\xi_j)}}}\right)$$

standard normal

$$P\left(\sum_{i=1}^{3} a_{ij} x_i \geq \xi_j\right) \geq p \quad \Leftrightarrow \quad \frac{\sum_{i=1}^{3} a_{ij} x_i - E(\xi_j)}{\sqrt{Var(\xi_j)}} \geq \underset{\uparrow}{q_p}$$

p-quantile of standard normal

Linear optimization problem:

$$\boxed{max \ \left\{ \sum_{i=1}^{3} a_{i,15} x_i \;\middle|\; \sum_{i=1}^{3} a_{ij} x_i \geq b_j + \underbrace{q_p \sqrt{Var(\xi_j)}}_{\text{safety term}} \ (j = 1,\ldots,15) \right\}}$$

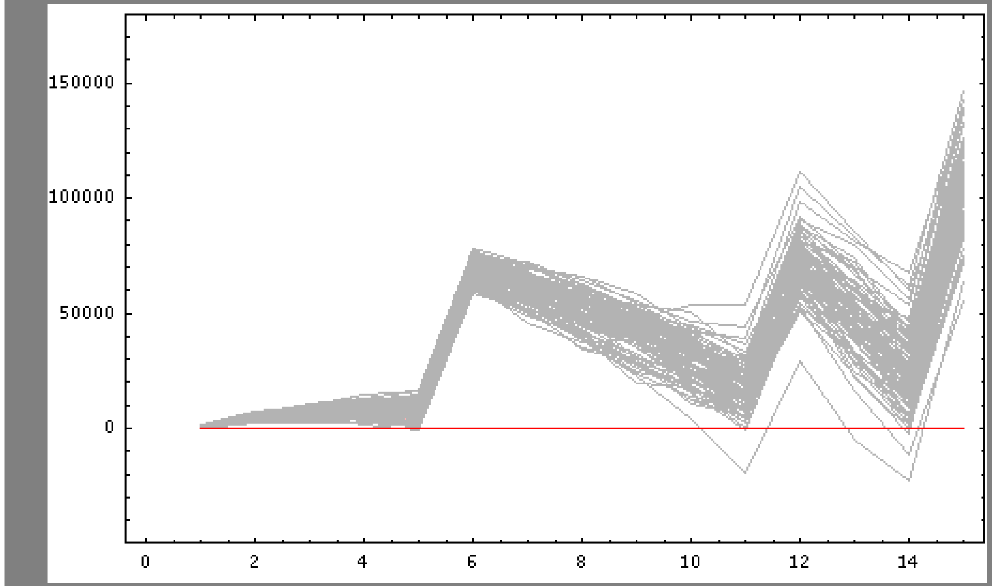
Solution of the cash matching problem with individual probabilistic constraints (p=0.95)

$$x^{opt}=\{62.8\,,72.6\,,101.1\}\ \Rightarrow\ f\left(x^{opt}\right)=103{,}925$$

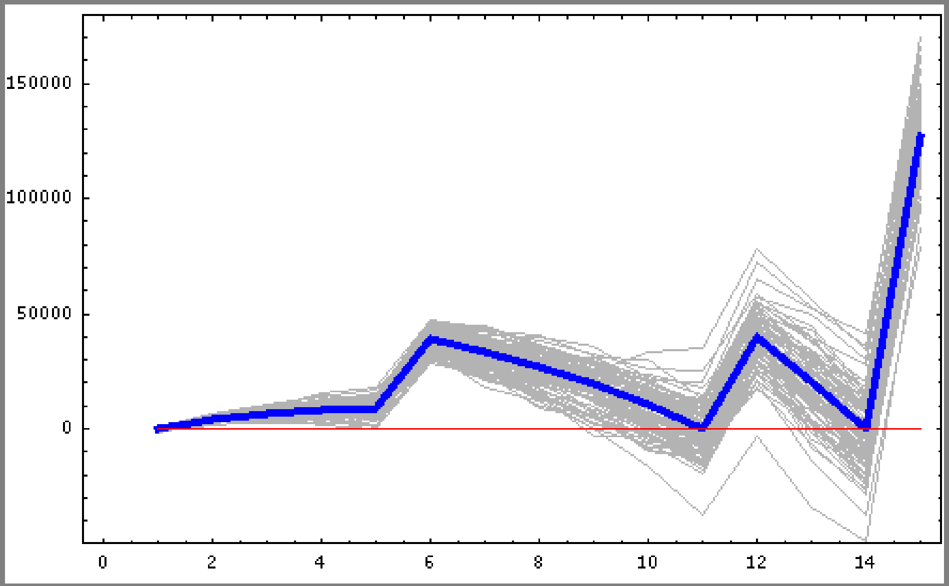
recall expected value (or deterministic) solution:

$$x^{opt}=\{31.1\,,55.5\,,147.3\}\ \Rightarrow\ f\left(x^{opt}\right)=127{,}332$$

cash (indiv. prob. cons.) 100 scenarios

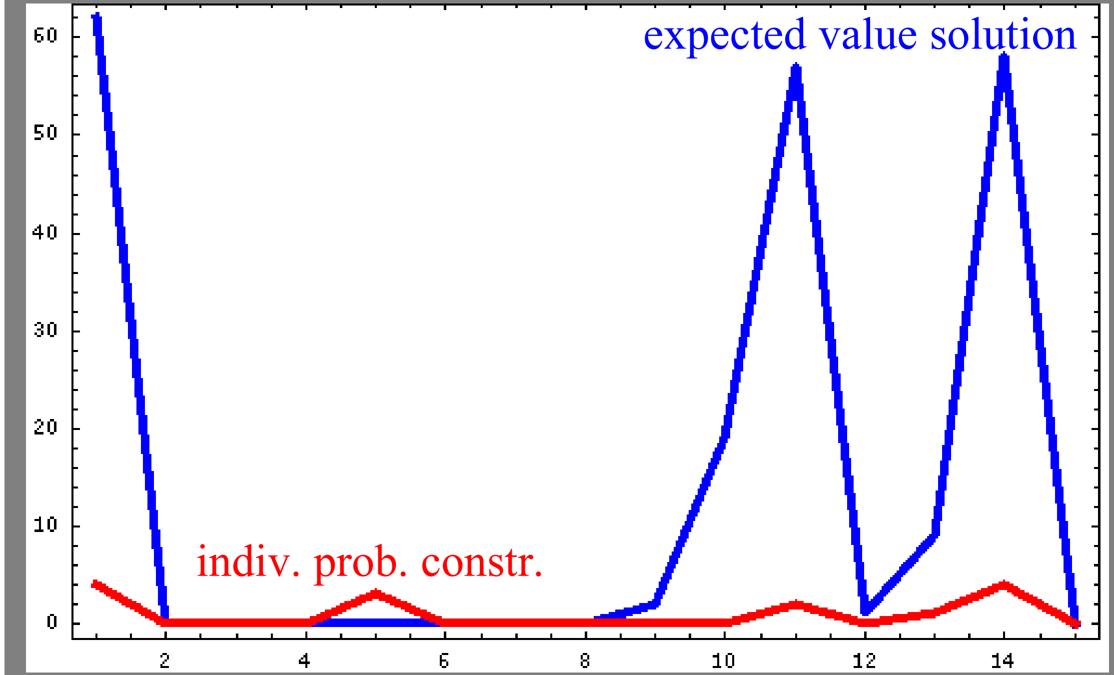


cash (expected value solution) 100 scenarios

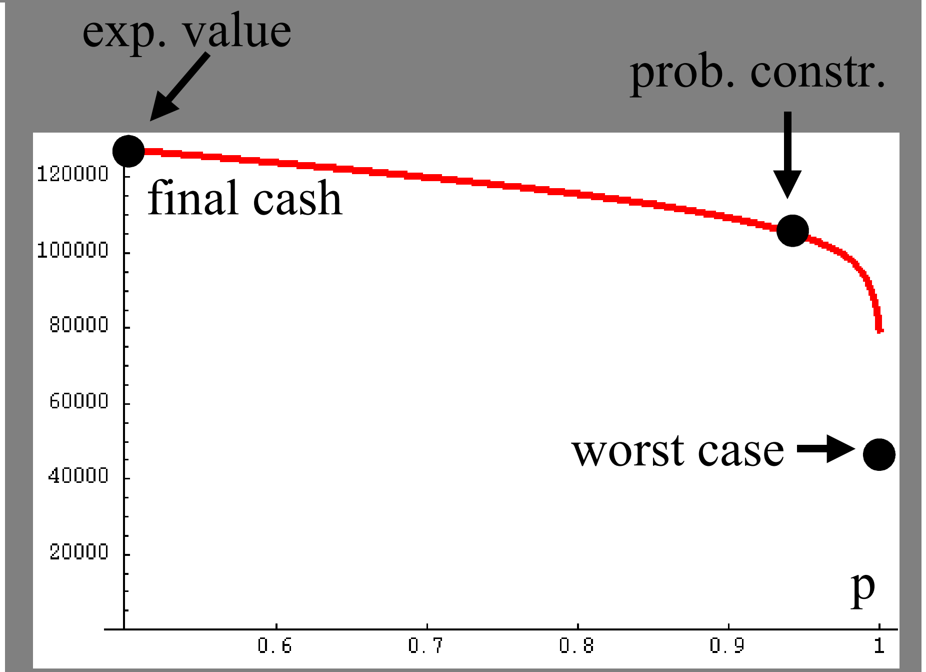


(continued)

number of scenarios violating positive cash (empirical probability of constraint violation)



final cash as a function of the probability level p



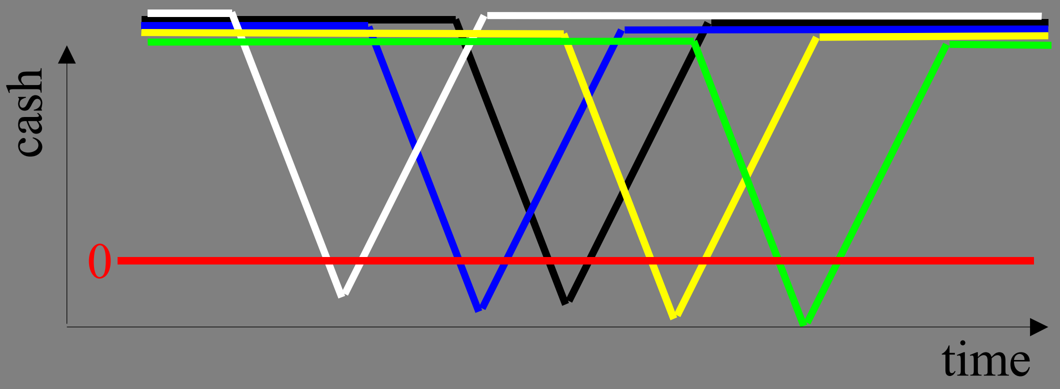
→ over a wide range robustness can be increased at moderate costs

Model with joint probabilistic constraints

individual probabilistic constraints:

at each fixed time probability of constraint violation low

but: over the whole time interval violation may be likely



at each t at most 1 out of 5 scenarios violates positive cash

but: all 5 scenarios violate positive cash

Therefore, replace the collection $$P\left(\sum_{i=1}^{3} a_{ij} x_i \geq \xi_j\right) \geq p \quad (j=1,\ldots,15)$$

of individual probabilistic constraints by one single

joint probabilistic constraint: $$P\left(\sum_{i=1}^{3} a_{ij} x_i \geq \xi_j \;\; (j=1,\ldots,15)\right) \geq p$$

Solution of the cash matching problem with joint probabilistic constraints (p=0.95)

$$x^{opt}=\{31.1\,,55.5\,,147.3\}$$

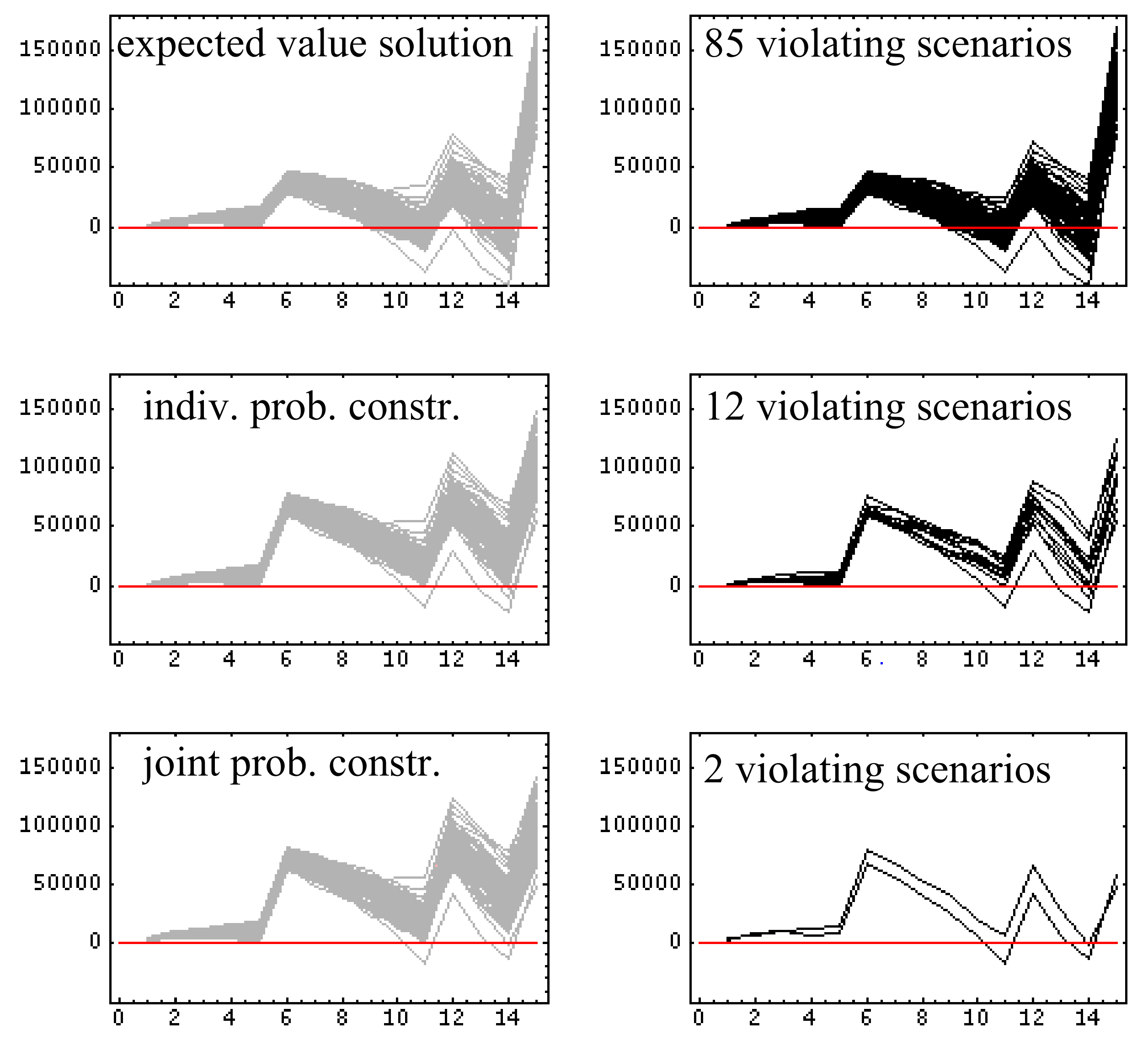
$$\Rightarrow \quad f\left(x^{opt}\right)=127{,}332$$

$$x^{opt}=\{62.8\,,72.6\,,101.1\}$$

$$\Rightarrow \quad f\left(x^{opt}\right)=103{,}925$$

$$x^{opt}=\{65.8\,,83.7\,,86.2\}$$

$$\Rightarrow \quad f\left(x^{opt}\right)=98{,}160$$



2. Models

Types of Probabilistic Constraints

Probabilistic constraints may be given in

individual: $P(h_j(x,\xi)\geq 0) \;\geq\; p \quad (j=1,\dots,m)$

or joint form: $P(h_j(x,\xi)\geq 0 \quad (j=1,\dots,m)) \;\geq\; p$

decision vector   random vector

cash matching problem: $h_j(x,\xi)=\sum_{i=1}^{3} a_{ij}x_i-\xi_j$

Structural properties and solution methods depend on:

- Form of the probabilistic constraint (individual or joint)
- Distribution of the random vector (continuous, discrete, independence)
- Properties of the constraint function h (linear, convex, separable)

Random right-hand side

Assume that the constraint function has separable structure:

$$h_j(x,\xi) \;=\; g_j(x) - \xi_j$$

individual prob. constr.:

$$P(h_j(x,\xi)\geq 0) \;\geq\; p \;\Leftrightarrow\; P(g_j(x)\geq \xi_j) \;\geq\; p \;\Leftrightarrow\; g_j(x) \;\geq\; \overset{\text{p-quantile}}{\overset{\downarrow}{q_p}} \quad (j=1,\dots,m)$$

Probabilistic constraints of same type as deterministic constraints

(no additional difficulties by randomness).

Cash matching problem: g_j linear ⟶ prob. constr. linear too

joint prob. constr.:

$$P(h_j(x,\xi)\geq 0 \;\; (j=1,\dots,m)) \;\geq\; p \;\Leftrightarrow\; P(g_j(x)\geq \xi_j \;\; (j=1,\dots,m)) \;\geq\; p$$

$$\Leftrightarrow\; \underset{\nearrow}{F_\xi}(g(x)) \;\geq\; p \qquad \text{(no quantile!)}$$

multivariate distribution function

g is analytically given, in general. Main task: calculate F_ξ

Independent components

joint prob. constr. with random rhs.: $F_{\xi}(g(x)) \;\geq\; p$

$\xi_1,\dots,\xi_s$ *independent* $\Rightarrow\; F_{\xi_1}(g_1(x))\cdots F_{\xi_s}(g_s(x)) \;\geq\; p$

1-dimensional distribution functions
(easy to calculate)

Independence of components rarely satisfied.

Cash matching problem: payments η_j were assumed to be independent

But: cumulative payments ξ_j are correlated nevertheless.

covariance matrix:

$$Var(\eta_j)=\sigma_j^2$$

$$\begin{pmatrix} \sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 & \cdots & \sigma_1^2+\sigma_2^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 & \cdots & \sigma_1^2+\sigma_{15}^2 \end{pmatrix}$$

Polyhedral Probabilistic Constraints

Let $h_j(x,\xi) = b_j(x) - \langle a_j(x), \xi \rangle \quad (b_j : \mathbb{R}^n \to \mathbb{R},\ a_j : \mathbb{R}^n \to \mathbb{R}^s)$

joint prob. constr.: $P\left(b_j(x) \geq \langle a_j(x), \xi \rangle \ (j = 1,\ldots,m)\right) \geq p, \quad (1)$

Decision and random vector no longer separated.

x fixed: probability of a polyhedron

Multivariate normal distribution: $\xi \sim N(\mu, \Sigma), \ a_j(x) \equiv a_j$

$\eta_j := \langle a_j, \xi \rangle \ \rightarrow \ (1): \ F_\eta(b(x)) \geq p, \quad \eta \sim N(A\mu, A \Sigma A^T)$

➡ normal probability of a polyhedron = normal distribution function value

But: If m>s (more inequalities than random components, e.g., networks)

➡ singular normal distribution

Random Coefficients

Let $h_j(x,\xi) = b_j(x) - \langle x, \xi_j \rangle \quad (b_j : \mathbb{R}^n \rightarrow \mathbb{R},\ \xi_j \in \mathbb{R}^n)$

Special case of polyhedral prob. constr.:

$a_j(x) \equiv x,\ \ \xi^T := (\xi_1^T, \ldots, \xi_m^T)$

joint probabilistic constraint: $P\left(\Xi x \leq b(x)\right) \geq p$

↑

random matrix

Example: cash matching problem with random yields.

3. Structure

Structural properties

We are interested in properties (continuity, differentiability, convexity) of

the constraint function $\alpha(x) \ := \ P(h_j(x,\xi) \geq 0 \ \ (j=1,\ldots,m))$

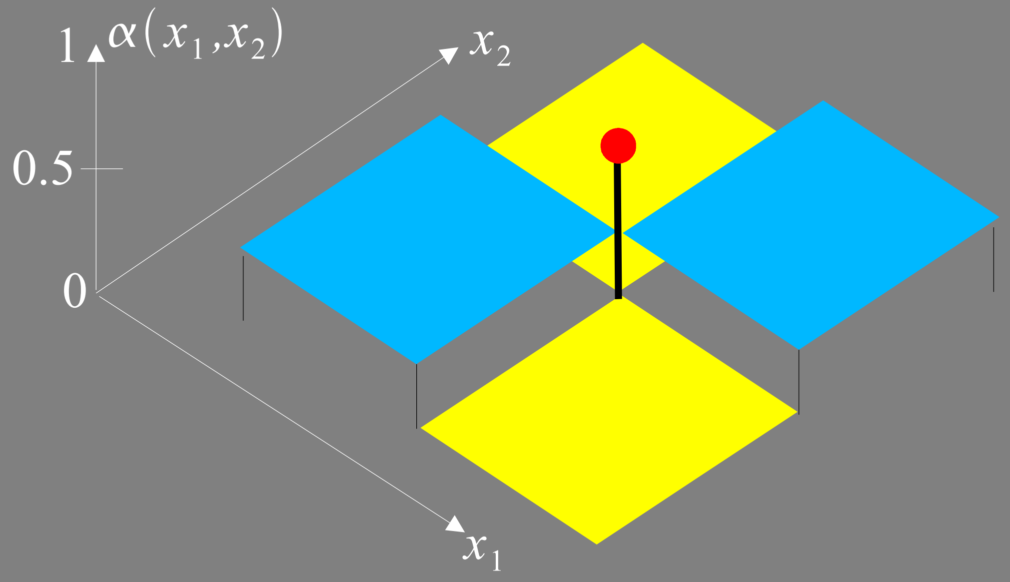
and of the feasible set $M \ := \ \{x \in \mathbb{R}^n | \alpha(x) \ \geq \ p\}$

Lemma: h_j *upper semicontinuous* $\Rightarrow$ α *upper semicontinuous* $\Rightarrow$ M *closed*

Example: $\xi \sim$ *standard normal*, $h_1(x,\xi) := x_1 \xi$, $h_2(x,\xi) := x_2 \xi$

$\Rightarrow$ α *discontinuous although all data are smooth* (h_j, *distribution of* ξ)

For 0<p<1 , the feasible set (blue) is not convex.



Properties by composition

recall joint prob. constr. with random rhs.: $F_\xi(g(x)) \geq p$

When is $F_\xi \circ g$ continuous, Lipschitzian, differentiable ?

By composition, all these properties are inherited from F_ξ, g.

This is obvious, in general, for g. How about F_ξ?

Lemma:

F_ξ is continuous if it has a density.

F_ξ is Lipschitzian if it has bounded marginal densities[1]

F_ξ is differentiable if the functions $t \rightarrow F_\xi(x|x_i = t)$ are continuous for all x,i.

conditional distribution function

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[1] Römisch/Schultz (1993)

Convexity

joint prob. constr. with random rhs.: $F_\xi(g(x)) \geq p \quad (2)$

When is $F_\xi \circ g$ concave ? ⟶ convex optimization algorithms

components g_j concave, A_1 concave and nondecreasing

e.g., linear, see
cash matching problem

automatically satisfied
for distribution functions

When is F_ξ concave ? Never!

(distribution functions are bounded from below and above but not constant)

Is there a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that $\varphi \circ F_\xi$ concave ?

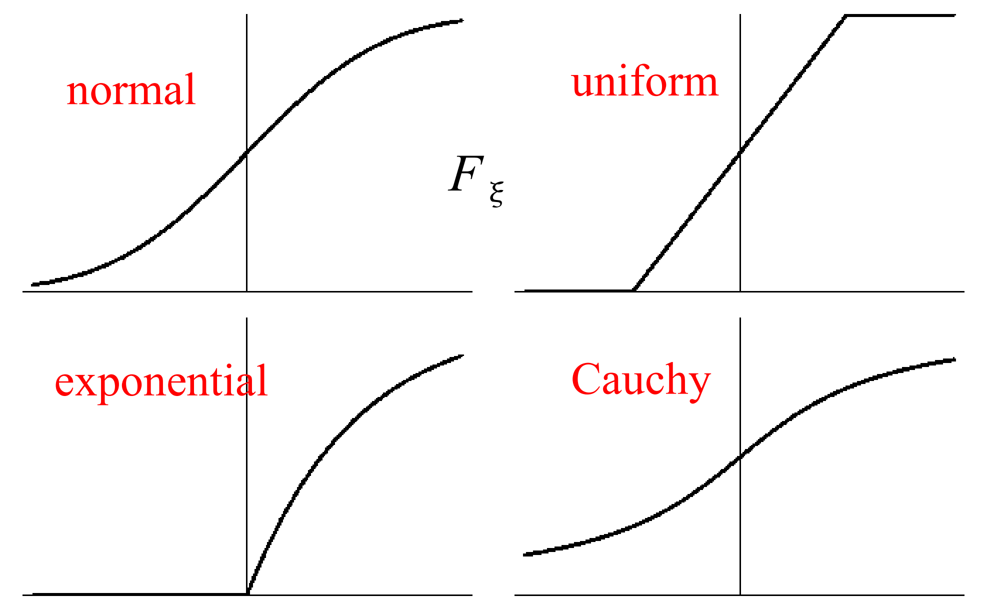
If so, then $(2) \Leftrightarrow \varphi(F_\xi(g(x))) \geq \varphi(p)$

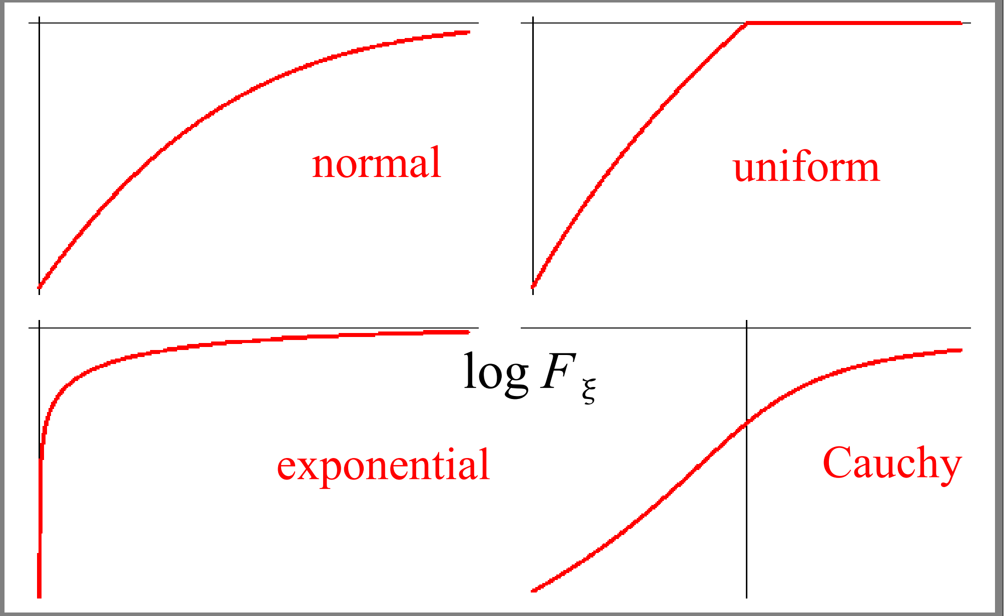
$(\varphi \circ F_\xi) \circ g$ concave if components g_j concave.

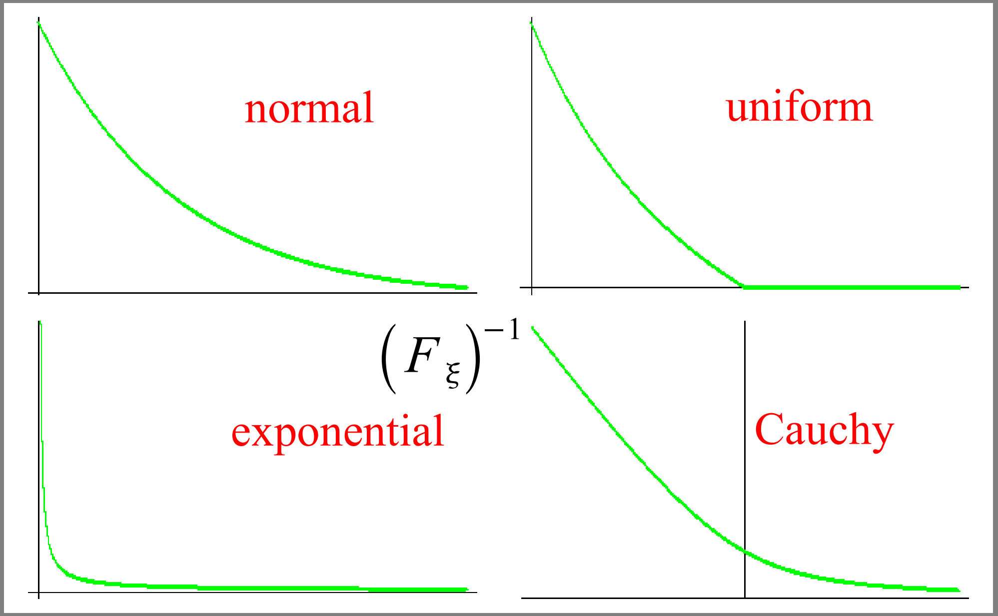
increasing and concave

Possible candidates: $\varphi = \log\,, \quad \varphi = -(\cdot)^{-n}$

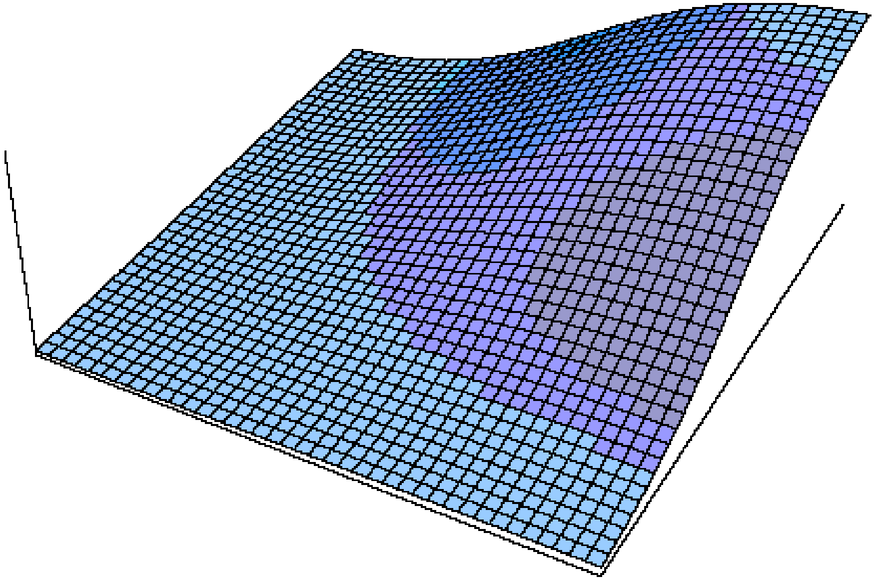
Logconcavity of distribution functions



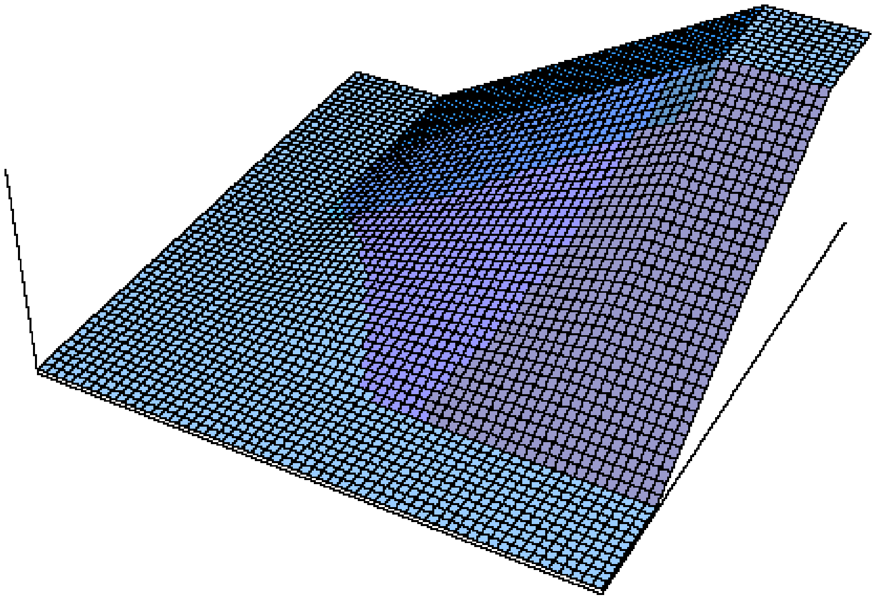




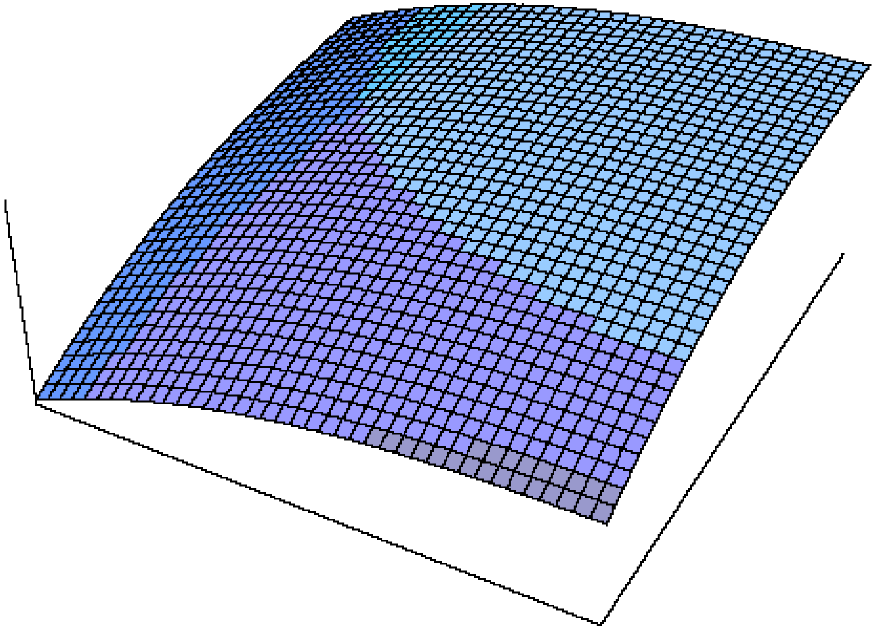
bivariate normal



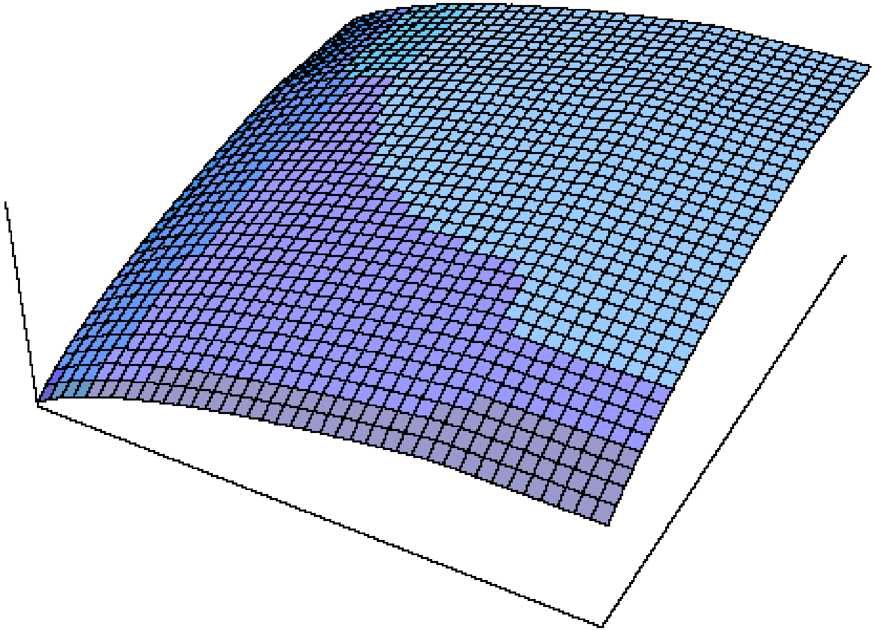
uniform on unit square



Log (bivariate normal)



Log (uniform on unit square)



Prekopa's Theorem (reduced version)

multivariate distribution functions extremely difficult to calculate

(no analytic formula)

almost impossible to check log-concavity directly

Prekopa's Theorem:

If the distribution function F_ξ has a density f_ξ, and if $\log f_\xi$ is concave, then $\log F_\xi$ is concave too.

Example: multivariate normal distribution

$$f_\xi = K \exp(-1/2 \ (x-\mu)^T \Sigma^{-1} (x-\mu))$$

$\log f_\xi = \log K - 1/2 \ (x-\mu)^T \Sigma^{-1} (x-\mu)$ concave $\longrightarrow$ $\log F_\xi$ concave

Other examples:

uniform distribution on convex compact sets, Dirichlet, Gamma, etc.

Van de Panne/Popp - Theorem
(extended version)

Theorem:

Let $\xi \sim N(\mu, \Sigma)$ *have a multivariate normal distribution.*

Consider a single probabilistic constraint with random coefficients:

$$\underbrace{P(\langle x, \xi \rangle \leq 0)}_{\alpha(x)} \geq p$$

The induced feasible set $M = \{x \in \mathbb{R}^n \mid \alpha(x) \geq p\}$ *is convex, if*

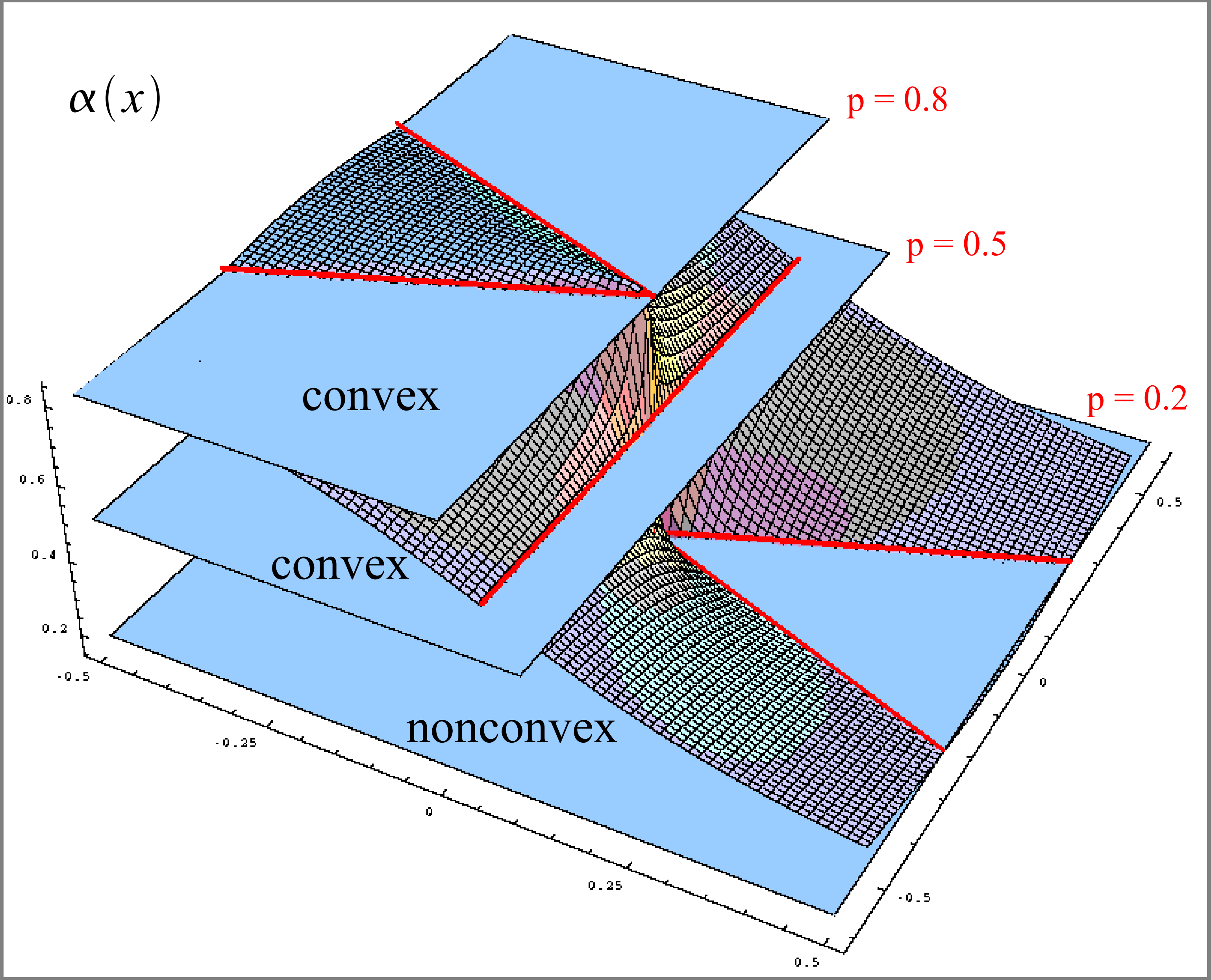
$$p \in [0, \Phi(-\lambda^{-1/2} \|\mu\|)] \cup [0.5, 1],$$

1-dim. standard normal distribution function

smallest eigenvalue of Σ

Theorem generalizes to several individual prob. constr. (intersection of convex sets is convex) but not to joint prob. constr. (see example above).

Example: $M = \{x \in \mathbb{R}^2 \,|\, \underbrace{P(\langle x, \xi \rangle \leq 0)}_{\alpha(x)} \geq p\}$ $\quad \xi \sim N((1,0), \boldsymbol{I}_2)$



Related Properties

constraint set $M = \{x \mid P(Ax \geq \xi) \geq p\}$

Theorem:[1] Let ξ have any distribution and let A have positively linear independent rows. Then, M is connected.

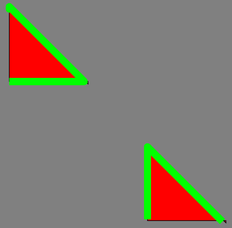
Example: ξ ~ uniform distribution on (1,-2,0),(-2,1,0),(-2,-2,0), p=0.5

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ M not convex but connected

rows pos. lin. indep.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$ M not connected

rows not pos. lin. indep.



cash matching problem does not satisfy the assumption!

Theorem:[2] Let ξ ~ nondeg. multiv. normal, independent components. Then, log F_ξ is strongly concave on bounded sets.

Let ξ ~ uniform on s-dim. interval [a,b]. Then, log F_ξ is strongly concave on int[a,b]. Not true for arbitrary polytopes!

1: H. (2001) 2: H./Römisch (2004)

4. Numerics

Random r.h.s, normal distribution

joint linear prob. constr. with random rhs.:

$$P(Ax \geq \xi) \geq p \Leftrightarrow F_{\xi}(Ax) \geq p$$

Assumptions ξ has a nondegenerate multivariate normal distribution

example: cash matching problem with joint probabilistic constraints

→ $F_{\xi} \circ A$ differentiable and concave (recall that F_{ξ} is log-concave)

→ Algorithms from differentiable convex optimization
(e.g., supporting hyperplanes)

numerical requirements: calculation of F_{ξ} and of ∇F_{ξ} .

calculation of gradients may be reduced to the calculation of values[1]
(in one dimension less)

→ main challenge: calculation of F_{ξ}

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[1] Prekopa: Stochastic Programming

Calculating the normal distribution function

(s = dimension of random vector)

- 'exact' calculation for s = 1,2[1,2]
- numerical integration (up to s~15)[3,4]
- efficient simulation techniques[5,6]
- bounds (linear programming[7]; graph-theoretical constructions[8]
- combined use of bounding and simulation techniques (up to s~50)

1: Donnely (1973) 2: Drezner, Wesolowsky (1990) 3: Schervish (1984) 4: Genz (1992) 5: Deak (1986)
6: Szantai (1985) 7: Prekopa (1990) 8: Bukszar, Prekopa (2001),

Bounds

Let ξ have a 4-dim. standard normal distribution (independent components)

For x:=(2,1,2,1), we want to estimate $F_\xi(x) = P(\xi_1 \leq 2, \, \xi_2 \leq 1, \, \xi_3 \leq 2, \, \xi_4 \leq 1)$

$\Phi := 1 - dim. \, standard \, normal \, distr. \, funct., \quad \Phi(1) = 0.841, \quad \Phi(2) = 0.977$

$\longrightarrow$ $exact \;\; value: \;\; F_\xi(x) = \Phi(2) \cdot \Phi(1) \cdot \Phi(2) \cdot \Phi(1) = 0.676$

trivial upper bound: $F_\xi(x) \leq min\{\Phi(1), \Phi(2)\} = 0.841$

$A_i := (\xi_i \geq x_i) \;\; (first \;\; order \;\; complementary \;\; events)$

$A_{ij} := (\xi_i \geq x_i \wedge \xi_j \geq x_j) \;\; (second \;\; order \;\; complementary \;\; events)$

$S_1 := \sum_i P(A_i), \;\; S_2 := \sum_{i<j} P(A_{ij})$ can be calculated exactly
(1- and 2-dim. normal distributions)

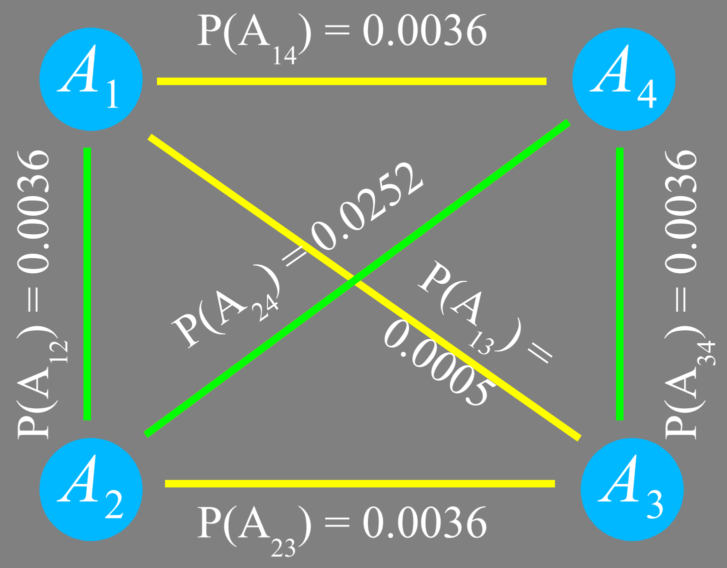
In the example: $S_1 = 0.363, \;\; S_2 = 0.040$

Bonferroni bounds: $1 - S_1 \leq F_\xi(x) \leq 1 - S_1 + S_2 \;\Rightarrow\; 0.637 \leq F_\xi(x) \leq 0.677$

improved lower bound: $1 - S_1 + (\frac{2}{s}) S_2 \leq F_\xi(x) \;\Rightarrow\; 0.657 \leq F_\xi(x)$

Hunters lower bound

Define a graph with nodes = A_i, edges = A_{ij} and weights of edges = $P(A_{ij})$:



Find a spanning tree T with maximum weight #T.

→ new lower bound: $1 - S_1 + \#T$

Here: $\#T = 0.0324$. →

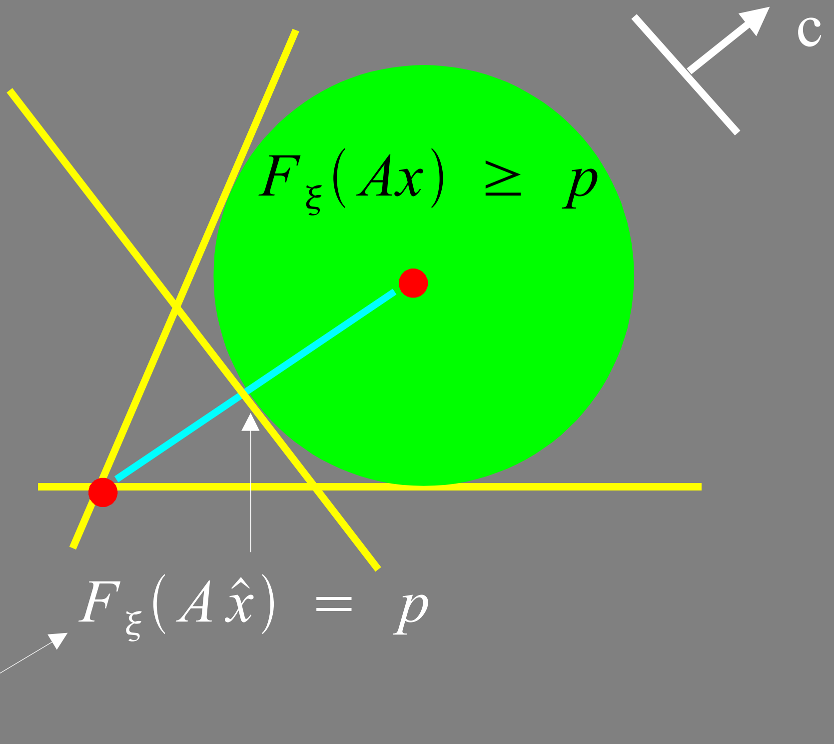
$$F_\xi(2,1,2,1) \ \geq \ 1 - 0.3628 + 0.0324 \ = \ 0.670$$

Comparison of bounds:

true value:	0.676
trivial upper bound:	0.841
Bonferroni lower bound:	0.637
Bonferroni upper bound:	0.677
improved lower bound:	0.657
Hunters lower bound:	0.670

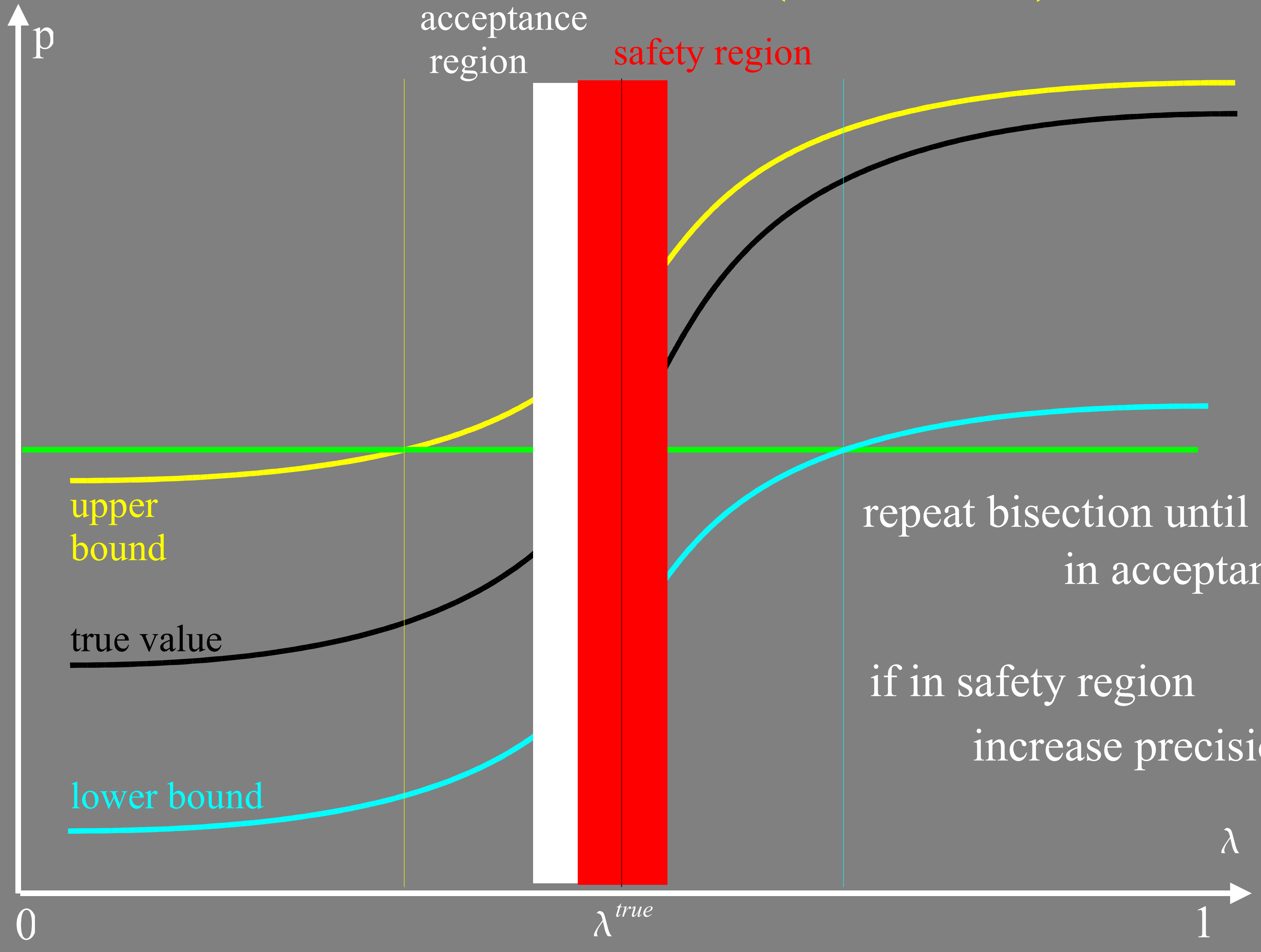
Cutting Plane Method

$$min\{\langle c,x\rangle | F_{\xi}(Ax) \geq p\}$$



line search: precise values expansive, imprecise values require safe cuts

Line Search (Szantai)



repeat bisection until
in acceptance region

if in safety region
increase precision

Discrete Distributions

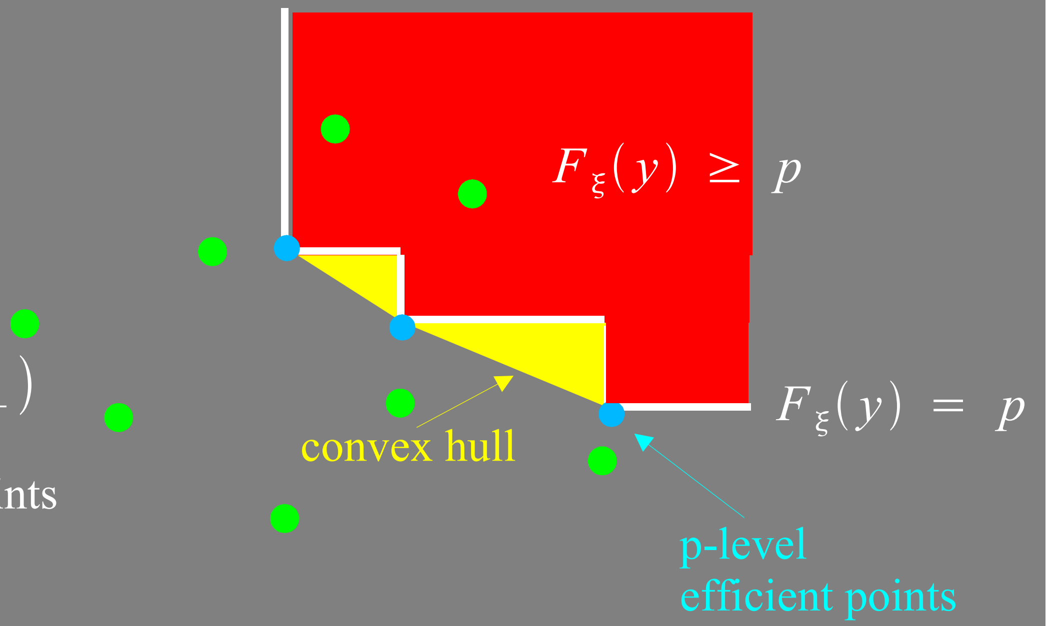
$$min\{\langle c,x\rangle | F_\xi(Ax) \ge p\} \quad (3)$$

$$\xi \sim (finite) \;\; discrete \;\; distribution$$

$$p-level \;\; set \;\; of \;\; F_\xi:$$

$$\{y \in \mathbb{R}^s | F_\xi(y) \ge p\} = \cup_{j=1}^N (z^j + \mathbb{R}^s_+)$$

p-level efficient points



$$(3) \;\Leftrightarrow\; min_{j\in J}\, min\{\langle c,x\rangle | Ax \ge z^j\}$$

conceptual solution method:

- determine all p-level efficient points (too many!)
- solve all resulting linear programs
- select the best solution (out of finitely many)

relaxed problem: $min\{\langle c,x\rangle | Ax \in co \;\{\cup_{j=1}^N (z^j + \mathbb{R}^s_+)\}\}$

$$\longleftrightarrow \quad min\{\langle c,x\rangle | Ax \ge \sum_{j=1}^N \lambda_j z^j,\;\; \lambda_j \ge 0,\;\; \sum_{j=1}^N \lambda_j = 1\}$$

stepwise generation of p-level efficient points[1]

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[1] Dentcheva, Prekopa, Ruszczynski (2000)

5. Stability

Approximation

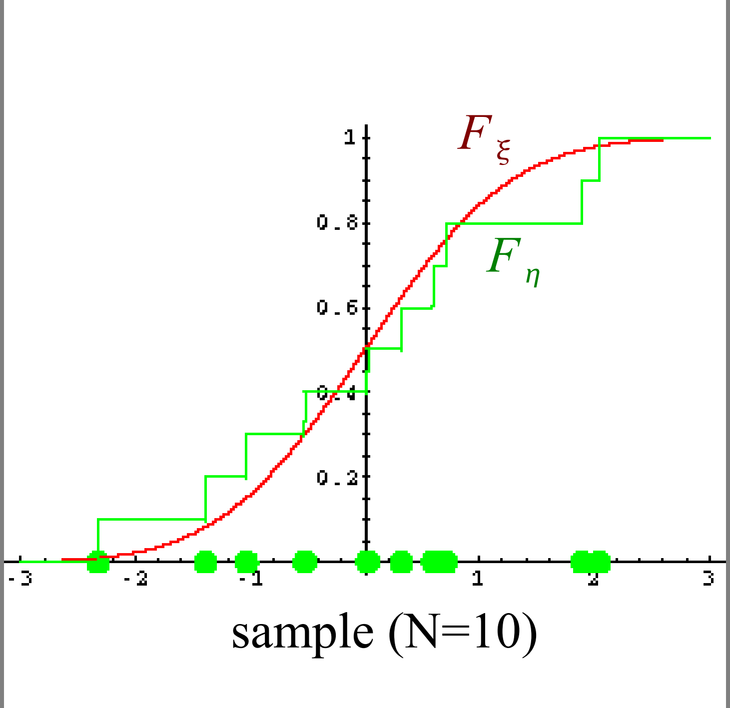
Original optimization problem with linear joint probabilistic constraint:

$$min\{f(x)|F_\xi(Ax)\ \geq\ p\}$$

Assumption: f convex, F_ξ log-concave (e.g., multivariate normal).

Problem: distribution of ξ in general not known

→ Approximation by some η (e.g., empirical approximation)



sample (N=10)

$$v(\eta)\ :=\ inf\{f(x)|F_\eta(Ax)\ \geq\ p\}$$

(optimal value)

$$S(\eta)\ :=\ \{x|F_\eta(Ax)\ \geq\ p,\ \ f(x)=v(\eta)\}$$

(solution set)

How do v and S behave near ξ?

Qualitative Stability

$$min\{f(x)|F_\xi(Ax) \geq p\} \quad (P)$$

Theorem:[1]

For the original problem (P) suppose that:

- $S(\xi)$ is nonempty and bounded
- There is some $\bar{x}$ with $F_\xi(A\bar{x}) > p$ (Slater point).

Then, S is upper semicontinuous at ξ (approximating solutions will converge to a true solution).

Furthermore, there are $L, \delta > 0$, such that

$$|v(\xi) - v(\eta)| \leq L\|F_\xi - F_\eta\|_\infty \quad \forall \eta: \|F_\xi - F_\eta\|_\infty < \delta$$

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[1] H./Römisch (2004)

Lipschitz-Continuity of the Value-at-Risk

$$VaR_p(\xi) = inf\{t \mid P(\xi \le t) \ge p\} \quad p \in [0,1]$$

usually not continuous!

Corollary:

Let $p \in (0,1)$ and ξ have a log-concave distribution .

Then, there exist $L, \delta > 0$ such that

$$|VaR_p(\xi) - VaR_p(\eta)| \le \|F_\xi - F_\eta\|_\infty \quad \forall \eta : \quad \|F_\xi - F_\eta\|_\infty < \delta$$

Quantitative Stability (Hölder type)

$$min\{\langle c,x\rangle+\langle x,Hx\rangle \mid F_\xi(Ax)\ \geq\ p\}\quad (H\geq 0)$$

convex quadratic objective

Theorem:[1]

In addition to the assumptions in the previous Theorem, suppose that F_ξ is strongly convex on some open convex set $U\supseteq A(S(\xi))$

Then, there are $L,\delta>0$, such that

$$d_H(S(\xi),S(\eta))\ \leq\ L\sqrt{\|F_\xi-F_\eta\|_\infty}\quad \forall\eta:\ \|F_\xi-F_\eta\|_\infty<\delta$$

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[1] H./Römisch (2004)